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Contents

Notations

Many of the notations we use are defined in the text where they appear for the first time. However, the following symbols and notations are used throughout this thesis.

Let N be a finite set. The power set of N is denoted by 2^N, i.e., $2^N := \{S \mid S \subset N\}$. Further, $\mathcal{P}_0(N)$ denotes the set of all non-empty subsets of N, $\mathcal{P}_0(N) := 2^N \setminus \{\emptyset\}$.
The set of real numbers is denoted by $\mathbf{R}$, and $\mathbf{N} := \{1, 2, 3, \ldots\}$ denotes the set of natural numbers. By $\mathbf{R}^N$ we denote the set of all functions from N to $\mathbf{R}$. The elements of $\mathbf{R}^N$ will be identified with $|N|$-dimensional vectors whose coordinates are indexed by the elements of N ($|N|$ denotes the cardinality of N). If $x \in \mathbf{R}^N$ and $i \in N$, we will write x_i instead of $x(i)$. If $x \in \mathbf{R}^N$, and $\emptyset \neq S \subset N$, we write x_S for the restriction of x to S, i.e., $x_S := (x_i)_{i \in S} \in \mathbf{R}^S$. $e^S \in \mathbf{R}^N$ denotes the vector with $e_i^S = 1$ if $i \in S$, and $e_i^S = 0$ otherwise. Instead of $e^{\{i\}}$ we simply write e^i. Further, we define $x(S) := \sum_{i \in S} x_i$.
For $x, y \in \mathbf{R}^N$, we write $x \geq y$ if $x_i \geq y_i$ for all $i \in N$, and $x > y$ if $x_i > y_i$ for all $i \in N$. Let $\mathbf{R}_+^N := \{x \in \mathbf{R}^N \mid x \geq 0\}$, and $\mathbf{R}_{++}^N := \{x \in \mathbf{R}^N \mid x > 0\}$. For $x, \lambda \in \mathbf{R}^N$, we define $\lambda * x \in \mathbf{R}^N$ by $(\lambda * x)_i := \lambda_i x_i$ for all $i \in N$. Further, $||x||$ denotes the Euclidean norm of $x \in \mathbf{R}^N$.
Let $A \subset \mathbf{R}^N$ and $\lambda \in \mathbf{R}^N$. Define $\lambda * A := \{\lambda * a \mid a \in A\}$. The convex hull of A is denoted by $conv(A)$. A is called comprehensive if $x \in A$ and $y \leq x$ imply $y \in A$. The comprehensive hull of A is the set $comp(A) := \{x \in \mathbf{R}^N \mid x \leq y$ for some $y \in A\}$.
Finally, the set of all permutations of N is denoted by $\Pi(N)$. For $x \in \mathbf{R}^N$ and $\sigma \in \Pi(N)$, we define $\sigma(x) \in \mathbf{R}^N$ by $\sigma(x)_i := x_{\sigma(i)}$ for all $i \in N$. $A \subset \mathbf{R}^N$ is called symmetric if for all $x \in A$, and all $\sigma \in \Pi(N)$, we have $\sigma(x) \in A$.

Chapter 1

Introduction

The subject of this thesis is the study of situations where a group of individuals with possibly conflicting interests is involved in a joint decision making process. We discuss mathematical models that have been developed in the literature to give more insight in several of these problems.

Part I studies situations where individuals work together in a project and where joint benefits or costs have to be allocated among the individuals. In the literature several methods are investigated to divide these joint benefits or costs. We pay special attention to so-called compromise solutions, which are based on the intuitively appealing principle that the amount allocated to each individual is obtained as a compromise between a pessimistic lower bound and an optimistic upper bound (which need not be the same for each individual).

In Part II we analyse situations in which there is one good that has to be divided among the individuals. It is assumed that every individual has single-peaked preferences, i.e., up to a certain point an individual likes to consume more of the good, beyond this point the opposite holds. Based on well-known results from cooperative bargaining theory, we provide two new characterizations of the most prominent division rule for these problems, the uniform rule.

Finally, Part III studies social choice problems. A central problem in social choice is, given the individual preferences of the agents over a certain set of alternatives, how to determine an outcome which is socially acceptable (reflects the individual preferences in a good way). It is interesting to study the collective power distribution induced by the possibilities of coalitions to manipulate the outcome which has to be chosen. A way to model this collective power distribution is by means of the concept of an effectivity function. In Part III we present several new results on effectivity functions.

The three parts of this thesis are self-contained, i.e., the thesis is written in such a way that one can read Part II or III without reading Part I. However, in order to avoid too much overlap between the three consecutive parts, for some basic notions we sometimes refer to a previous part where they were introduced.
In the next three sections we will give an introduction and overview of each of the three different parts. Each section starts with an example to give the reader an idea of the type of problems that are examined in the corresponding part.

1.1 Introduction to Part I

We consider a situation where there are three breweries, situated closely together.[1] The three firms conduct the transport to and from their clients themselves. In practice, it turns out that a truck which has delivered its goods at some of the clients, comes back to the brewery empty, while it would have been possible to pick up some retour cargo at another client in the neighbourhood. As a consequence of these inefficient matchings between trucks on the one hand and cargo on the other hand the transportation costs are much higher than necessary. For this reason one of the three brewers has developed a system which provides him better matchings between trucks and cargo. Moreover, by centralizing the transport from a separate depot nearby the brewery, a better overview over the transportation business is obtained. As a result of the reorganization of the transport, this brewery has realized a substantial decrease in the transportation costs.
However, this brewer realizes that still higher cost savings can be obtained if he cooperates with the other breweries by a joint use of his system. The reason for these extra savings lies mainly in the fact that better matchings between trucks and cargo are possible. An important question which arises is how to divide the joint cost savings among the three breweries if they work together. One way to solve this problem is to allocate the total savings proportional to the individual kilometre reductions. Clearly, the brewer who developed the system will not be satisfied with this allocation method. One obvious reason for him to reject this method is that it does not take into

[1]This example is derived from Borm, van Os, and Otten (1994).

account the fact that he has made a substantial investment to develop the system. This problem would probably not be very difficult to solve, but another reason for his dissatisfaction is the fact that he is in a stronger, or more powerful, position than the other two brewers. Without the use of his system no savings can be obtained or, formulated otherwise, his cooperation is necessary to save transportation costs (it is assumed that it would take a couple of years for the other breweries to develop a comparable system). The observation that the brewer who developed the system is in a more powerful position than the others indicates that the part of the joint savings that he will consider to be his 'fair share' will be more than his proportional share of the individual kilometre reductions. But how should we determine a 'fair' allocation?

A mathematical framework to analyse this type of problems is provided by cooperative game theory. The basis of cooperative game theory was laid in the fundamental book *"Theory of Games and Economic Behavior"* by von Neumann and Morgenstern in 1944. Since then, several models of cooperative games have been introduced and studied. In general, cooperative game theory deals with situations in which several parties (called players) are involved in a joint project and the (joint) benefits of this cooperation have to be shared. It is assumed that the players are *rational*, i.e., each player wants to maximize his own utility, being independent of the utilities of the other players (this is a standard assumption in economic theory).

A typical characteristic of game theory is that it does not study a single problem on its own, but that it embeds a specific problem in a whole class of problems with the same (mathematical) structure. The advantage of this approach is that, by studying a whole class of similar problems, solution methods can be developed and compared on a normative basis by investigating several (desirable) properties. For applications these comparisons can help to determine which solution method(s) fit(s) best to specific criteria demanded by concrete situations.

The approach of determining 'good' solution methods by means of required properties is called 'the axiomatic approach'. Sometimes a list of desirable properties for solution methods leads to a unique method which satisfies all listed properties. In this case we say that this solution method is 'characterized' (or axiomatized) by this list of properties.

Prominent models in cooperative game theory are transferable utility (TU) games, non-transferable utility (NTU) games, and bargaining problems. In TU-games it is assumed that interpersonal comparisons of utility are possible in the sense that the players have a common measure of utility (money), which can be freely transferred between the agents. NTU-games form a larger class of games in which there is not necessarily such a common measure of utility. Bargaining problems can be regarded as special NTU-games in which only the opportunities of the individual players and of the grand coalition are specified.

For each of these models several allocation methods (solution concepts) have been introduced. Well-known examples are the core and the Shapley value for TU-games and NTU-games, and the Nash solution for bargaining problems. In Part I of this thesis we focus attention on a special type of allocation methods, which are called compromise values. The basic principle behind compromise values is that the amount allocated to each participant should be determined as a compromise between an (optimistic) upper bound and a (pessimistic) lower bound. These bounds are sometimes interpreted as the utopia outcome and the minimal right of the participants.

In Chapter 2 we present a detailed overview of compromise values in cooperative game theory. Most attention is paid to the study of the τ-value for TU-games, the Raiffa-Kalai-Smorodinsky (RKS) solution for bargaining problems, and the compromise value for NTU-games. Chapter 2 is concluded with a number of applications of cooperative game theory, which clearly reflect the significance of the idea behind compromise values.

An important application of cooperative game theory is the systematic study of cost allocation problems. In Chapter 3 we study cost allocation problems in a game theoretic framework. Most attention is paid to a cost allocation method introduced by the Tennessee Valley Authority (TVA) in the 1930's. This method, called the alternate cost avoided method or ACA-method, was used by the TVA to solve a cost allocation problem generated by the building of a dam in the Tennessee river. Although the principle behind the ACA-method is different from the ideas behind the τ-value, it turns out that there is a strong similarity between both solution concepts. For a large subclass of cost allocation problems both methods prescribe the same outcome. Based on results for the τ-value we obtain new results for the ACA-method.

Chapters 4 and 5 deal with the study of solution concepts for NTU-games. Chapter 4 discusses the compromise value for NTU-games in more detail. This solution concept is given this general name because it extends both the τ-value for TU-games and the RKS-solution for bargaining problems to a large class of NTU-games. We extend

known results for the compromise value and moreover, new characterizations of this solution concept are obtained.
The observation that the Shapley value for TU-games can be regarded as a compromise solution, leads to the introduction of a new solution concept in Chapter 5. The so-called marginal based compromise value, or shortly MC-value, extends the Shapley value towards a subclass of NTU-games. Surprisingly, it can also be seen as an extension of the RKS-solution for bargaining problems towards NTU-games. We study several properties and provide two characterizations of the MC-value.

1.2 Introduction to Part II

Consider a situation where the parents of a little child have an appointment. This appointment will take a whole day, and they do not want to take the child with them. Fortunately, they have found four persons who are each willing to baby-sit for a certain period of the day. It is assumed that each of the four persons has in principle the whole day available for the baby, but they all have a preferred length of time to baby-sit, not necessarily being equal for all of them. The preferences of a person increase before this preferred length of time, and after this point the opposite holds. However, the four optimal points need not be compatible. They might add up to more or less than the total amount of time needed to baby-sit. The problem of interest is how to determine a 'fair division', or a fair amount of time to baby-sit for each of the four persons. In the literature this type of problem is called a 'fair division problem with single-peaked preferences'.

Single-peaked preferences often occur in practice. Everybody can imagine situations where preferences over a commodity become satiated at a certain point. An example in exchange economies being discussed by Sprumont (1991) is rationing at disequilibrium prices in a two-good economy. If preferences of the agents are strictly convex, then their restrictions to the budget lines are single-peaked.
The first one who analysed this fair division problem with single-peaked preferences in a systematic way is Sprumont (1991). He followed an axiomatic (or game theoretic) approach (cf. Section 1.1) to study well-behaved procedures (rules), that assign to each problem of this type an unique allocation. His main result establishes the existence of an unique Pareto optimal, anonymous, and strategy-proof allocation rule, which he called the uniform rule. The paper by Sprumont started an extensive

analysis of this type of problems. As a result of this analysis the uniform rule is now considered to be the most important rule for fair division problems with single-peaked preferences.

In Part II of this thesis we concentrate on the uniform rule. In Chapter 6 we formally introduce the model and the uniform rule. Further, we present an overview of the most important results in the literature.

In Chapter 7 we establish relations between the uniform rule, the Nash solution, and the lexicographic egalitarian solution for bargaining problems. Based on well-known properties and characterizations of these two solutions we introduce new properties which are satisfied by the uniform rule. Further, we show that the uniform rule can be characterized by means of these properties. Chapter 7 concludes with an application of our results to a cost-sharing model.

1.3 Introduction to Part III

Consider a situation in which the board of directors of a company, consisting of five persons, has to decide about the firm's long term policy. In the foregoing years this item has been the subject of a thorough study and several alternatives have been proposed by advisors of the company. After a careful comparison of the different options there are three alternatives remaining, and now it is time for the board to make a final decision among one of these alternatives. The first two alternatives do not have large consequences for the employees of the company, since it is not necessary to dismiss employees. However, the third option is more rigorous: A complete division is disposed of and consequently, several employees have to be dismissed.

The procedure that the board of directors follows to decide which alternative will be chosen is the following. Each member may state his most preferred option and then the alternative is chosen which is stated by most persons. However, since the third option has huge consequences for the company, it is agreed that this option is chosen only if there are at least four of the five members that favour this alternative.

We are interested in the opportunities that individual members or subgroups of the board have if they cooperate with other members and make agreements on how to vote in order to achieve a certain outcome. For this simple example it is not difficult to specify the power that subgroups (coalitions) have if they cooperate. No member of the board has the power to force a single alternative to be the outcome. However, if two members of the board cooperate, then they still cannot force an alternative to be the outcome, but they do have the power to prevent that the third option is chosen. We say that each 2-person coalition can *veto* the third option. Analogously, we see that every subgroup of three members has the power to force the first or second alternative as the outcome, but it can also veto every option. Moreover, coalitions consisting of four or five members have the power to veto or to choose every alternative.

As the previous example illustrates a collective decision rule induces a power distribution in the society. From a theoretical viewpoint it is interesting to study the power distribution induced by a decision rule because it gives insight in the possibilities that coalitions have available to manipulate the decision rule in order to obtain a more preferred alternative. A way to model the power distribution in society has been introduced in Moulin and Peleg (1982), using the concept of an effectivity function. An effectivity function specifies for each coalition the collection of subsets of alternatives for which the coalition is effective in the sense that it can force the outcome to be an element of such a subset of alternatives.

In Part III of this thesis we study effectivity functions. Chapter 8 formally defines the concept of an effectivity function and presents an overview of some of the main results that have been achieved in this area. Among others we present theorems of Moulin and Peleg (1982) and Moulin (1983) which illustrate relations between effectivity functions and non-cooperative game forms. Further, we consider three special classes of effectivity functions that play a prominent role in the literature: effectivity functions corresponding to monotonic simple games, additive effectivity functions, and effectivity functions corresponding to veto functions.

In Chapter 9 we introduce a new class of effectivity functions, called decomposable effectivity functions, which incorporates the three classes mentioned above. We study relations between properties of a decomposable effectivity function and properties of the pair of TU-games that generate this effectivity function. Moreover, two characterizations of the class of decomposable effectivity functions are provided.

Chapter 10 studies relations between effectivity functions and game correspondences, which generalize game forms. Our main results provide counterparts of the theorems of Moulin (1983) and Moulin and Peleg (1982) for larger classes of effectivity functions.

Part I

Compromise values for cooperative games

Chapter 2

Compromise values: an overview

Since the introduction of cooperative games by von Neumann and Morgenstern in 1944, the problem most extensively studied in cooperative game theory has been how to divide the total benefits of the grand coalition if all players cooperate. Many solution concepts have been proposed to handle this problem. Well-known examples are the core, the Shapley value and the nucleolus in games with transferable utility (TU-games), the core, the Shapley NTU-value and the Harsanyi value in non-transferable utility games (NTU-games), and the Nash bargaining solution in cooperative bargaining theory.

A special class of solution concepts is provided by so-called compromise values. The idea underlying this type of solution concepts is that the outcome of a game should be determined as an efficient compromise between an optimistic upper value and a pessimistic lower value, sometimes representing the utopia payoffs and the minimal rights of the players, respectively. Prominent examples of compromise values are the τ-value for TU-games, the compromise value for NTU-games, and the Raiffa-Kalai-Smorodinsky solution for bargaining problems.

This chapter, which is based on Tijs and Otten (1993), presents an overview of well-known compromise values for several cooperative models.

In Section 2.1 we recall some basic definitions and solution concepts for TU-games. Most attention is paid to the τ-value introduced by Tijs (1981). The τ-value plays a central role in Section 2.2, where we discuss several properties and characterizations of the τ-value.

In Section 2.3 we consider bargaining problems. Particularly, we are interested in the Raiffa-Kalai-Smorodinsky solution introduced by Raiffa (1953) and characterized by

Kalai and Smorodinsky (1975).
Section 2.4 is devoted to compromise values for NTU-games. We discuss two extensions of the τ-value to NTU-games introduced by Borm et al. (1992), namely the compromise value for NTU-games and the NTU τ-value.
In Section 2.5 we consider several applications of cooperative game theory in economics and operations research, and compare the outcomes prescribed by compromise values to outcomes of other economic or game theoretic solution concepts. The following applications are discussed: bankruptcy problems, big boss games, exchange markets, weighted graph games, and sequencing games. Another important application of cooperative game theory is the analysis of cost allocation problems. This is the subject of Chapter 3.

2.1 TU-games

In this section we examine compromise values for TU-games. We start with some basic definitions.

A *transferable utility game* or *TU-game* (von Neumann and Morgenstern (1944)) is an ordered pair (N, v) where N is a finite set of *players* and $v : 2^N \to \mathbf{R}$ is a map assigning to each *coalition* $S \in 2^N$ a real number $v(S)$, called the *worth* of S, and where $v(\emptyset) := 0$. Often, a TU-game (N, v) will be identified with the function v. The class of all TU-games with player set N is denoted by G^N, and by G we denote the class of all TU-games (with finite player set).

Example 2.1.1 We consider the situation of the three breweries described in Section 1.1. To model this situation as a TU-game, we first have to define the player set. In this case the players are the three breweries, which for the sake of convenience are numbered 1, 2, and 3 (1 represents the brewery which developed the system). Hence, $N = \{1, 2, 3\}$. The function $v : 2^N \to \mathbf{R}$ assigns to each coalition S the cost savings $v(S)$ that can be attained when the breweries in S cooperate on the transportation business. It is assumed that breweries 2 and 3 cannot accomplish significant cost savings without the presence of 1, because they cannot make use of his system. Thus, $v(S) = 0$ if $1 \notin S$. The cost savings of the different coalitions are summarized in Table 1.1.

coalition S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
savings $v(S)$	0	4	0	0	6	7	0	10

Table 1.1. The cost savings of the different coalitions (in \$ 1,000,000).

A TU-game v is called *convex* (Shapley (1971)) if for all coalitions $S, T \in 2^N$

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T).$$

Equivalently, a game v is convex if and only if for all $S, T \in 2^N$, and all $i \in N$ such that $S \subset T \subset N \setminus \{i\}$ we have

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T).$$

So for convex games the marginal contribution of a player to a large coalition is higher than to a smaller coalition. Notice that the game in Example 2.1.1 is convex.

One of the main topics dealt with in cooperative game theory is, given a game v, to divide the amount $v(N)$ between the players if the grand coalition N is formed. A *payoff vector* for v is a vector $x \in \mathbf{R}^N$ which is *efficient*, i.e., for which $\sum_{i \in N} x_i = v(N)$. Here x_i represents the payoff to player $i \in N$. A payoff vector $x \in \mathbf{R}^N$ is called an *imputation* if $x_i \geq v(\{i\})$ for all $i \in N$. The set of all imputations of the game v is denoted by $I(v)$. The *core of* v is the set

$$C(v) := \{x \in I(v) \mid \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in 2^N\}.$$

So, if $x \in C(v)$, then no coalition S has an incentive to split off if x is proposed as a payoff vector, because the total amount $x(S) := \sum_{i \in S} x_i$ allocated to S is not smaller than the amount $v(S)$ which they can obtain by forming a subcoalition.
The notion of the core has first been explicitly formulated and named in Gillies (1953), but the principle behind this concept was already known before (see for example Ransmeier (1942)). Also von Neumann and Morgenstern (1944) considered this concept, but at that stage they rejected it because the core of a game can be empty. However, there are large subclasses of TU-games which have a non-empty core. One example is the class of convex games as shown by Shapley (1971). Games with a non-empty core are called *balanced*.

Since the introduction of TU-games many solution concepts have been proposed to allocate the amount $v(N)$ among the players. Formally, a *solution concept* on a class $A \subset G$ is a map which assigns to each TU-game $(N, v) \in A$ a vector in $\mathbf{R}^N$ or a set of vectors in $\mathbf{R}^N$. The imputation set and the core are examples of (multivalued) solution concepts. Also many one-point solution concepts, which assign to a game v a unique vector, have been proposed. A one-point solution concept is also called a *rule* or a *value*. The most well-known values are the Shapley value introduced by Shapley (1953) and the nucleolus introduced by Schmeidler (1969).

The *Shapley value* $\Phi(v) \in \mathbf{R}^N$ of a game $v \in G^N$ is a weighted average of the marginal contributions of players to coalitions. Formally, the Shapley value of v is defined by

$$\Phi_i(v) := \sum_{S \subset N \setminus \{i\}} \frac{|S|!(|N|-1-|S|)!}{|N|!}(v(S \cup \{i\}) - v(S)) \text{ for all } i \in N.$$

Another way to introduce the Shapley value is based on marginal vectors. We order the players in a game $v \in G^N$ with $|N| = n$ by means of a bijection $\sigma : \{1, \ldots, n\} \to N$. With minor abuse of terminology we identify the class of all such bijections with the class $\Pi(N)$ of all permutations of N. For $i \in N$ and a permutation $\sigma \in \Pi(N)$, we define the set $P_\sigma(i) := \{j \in N \mid \sigma^{-1}(j) < \sigma^{-1}(i)\}$ as *the set of predecessors of i in σ*. The *marginal vector* $m^\sigma(v) \in \mathbf{R}^N$ is defined by

$$m^\sigma_{\sigma(i)}(v) := v(P_\sigma(\sigma(i)) \cup \{\sigma(i)\}) - v(P_\sigma(\sigma(i)))$$

for all players $\sigma(i) \in N$.

The vector $m^\sigma(v)$ assigns to each player his marginal contribution in the order σ. It is not difficult to show that the Shapley value of v is the average of all marginal vectors, i.e.,

$$\Phi(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} m^\sigma(v).$$

The nucleolus is a rule for the class of games with non-empty imputation set. Let $v \in G^N$ with $I(v) \neq \emptyset$ and let $x \in \mathbf{R}^N$ and $S \in 2^N$. The *excess* $E^v(S, x)$ *of S with respect to x* is defined as

$$E^v(S, x) := v(S) - x(S).$$

$E^v(S, x)$ measures the complaint of the coalition S against the proposal x.

Let $\Theta(x)$ be the $2^{|N|}$-tuple whose components are the excesses $E^v(S, x)$, $S \subset N$, arranged in a nonincreasing order, i.e., $\Theta_i(x) \geq \Theta_j(x)$ whenever $1 \leq i < j \leq 2^{|N|}$. $\Theta(x)$ is the *excess vector (complaint vector)* of x. The *nucleolus* $n(v)$ of v is the set of all imputations $x \in I(v)$ satisfying

$$\Theta(x) \leq^l \Theta(y) \text{ for all } y \in I(v),$$

where $\leq^l$ denotes the *lexicographic order* on $\mathbf{R}^{2^{|N|}}$: For $a, b \in \mathbf{R}^{2^{|N|}}$, $a \leq^l b$ if and only if $a = b$ or there exists a $k \in \{1, \ldots, 2^{|N|}\}$ such that $a_i = b_i$ for all $i \in \{1, \ldots, k-1\}$ and $a_k < b_k$. So the nucleolus has the property that it minimizes the maximal complaint. Schmeidler (1969) proved that the nucleolus of a game always consists of one point.

A third value is the τ-value as introduced by Tijs (1981) for quasi-balanced games. The τ-value of a game is a compromise between an upper and a lower value for the game. Let $v \in G^N$ be a TU-game. The vector $M(v) \in \mathbf{R}^N$ with coordinates

$$M_i(v) := v(N) - v(N \setminus \{i\})$$

for all $i \in N$ is called the *upper value of* v. $M_i(v)$ can be regarded as the maximal payoff player i can expect to get: If he claims more, then it is advantageous for the other players to exclude him from the grand coalition. $M_i(v)$ is also called the *utopia payoff* for player i.
Let $i \in N$ and $S \in 2^N$ with $i \in S$. We calculate what remains for player i if S forms and all other players in S obtain their utopia payoff. The *remainder of* $i \in S$, $R^v(S, i)$, is defined by

$$R^v(S, i) := v(S) - \sum_{j \in S \setminus \{i\}} M_j(v).$$

The vector $m(v) \in \mathbf{R}^N$ with coordinates

$$m_i(v) := \max_{S: i \in S} R^v(S, i)$$

for all $i \in N$ is called the *lower value of* v. The amount $m_i(v)$ denotes the *minimal right* of player i. He can guarantee himself this payoff by offering the members of a suitable coalition S, for which the maximum is achieved, their utopia payoff and then $m_i(v)$ remains for himself.
A game $v \in G^N$ is called *quasi-balanced* if

$$m(v) \leq M(v) \text{ and } \sum_{i \in N} m_i(v) \leq v(N) \leq \sum_{i \in N} M_i(v).$$

The class of all quasi-balanced games with player set N is denoted by QB^N. The fact that balanced games are quasi-balanced follows from the following theorem by Tijs and Lipperts (1982).

Theorem 2.1.2 Let $v \in G^N$, with $C(v) \neq \emptyset$. Then for all $x \in C(v)$, we have

$$m(v) \leq x \leq M(v).$$

For a game $v \in QB^N$ the *τ-value of* v, denoted by $\tau(v)$, is the unique payoff vector on the line segment with endpoints $m(v)$ and $M(v)$. Thus,

$$\tau(v) := m(v) + \alpha(M(v) - m(v)),$$

endpoints are not M(v),

where α is such that $\sum_{i \in N} \tau_i(v) = v(N)$.

Example 2.1.3 Let (N, v) be the 3-person game of Example 2.1.1.
Then $M(v) = (10, 3, 4)$, $m_1(v) = \max\{v(\{1\}),\ v(\{1,2\}) - M_2(v),\ v(\{1,3\}) - M_3(v),\ v(N) - M_2(v) - M_3(v)\} = \max\{4, 3, 3, 3\} = 4$, $m_2(v) = 0$, and $m_3(v) = 0$.
It follows that

$$\tau(v) = (4, 0, 0) + \alpha(6, 3, 4),$$

where α is such that $\sum_{i \in N} \tau_i(v) = 10$. Hence, $\alpha = \frac{6}{13}$ and $\tau(v) = \frac{1}{13}(88, 18, 24)$.
Note that for this game $\Phi(v) = \frac{1}{6}(41, 8, 11)$ and $n(v) = (6\frac{1}{2}, 1\frac{1}{2}, 2)$.
One easily verifies that in this case the τ-value, the Shapley value and the nucleolus all belong to the core.

Theorem 2.1.2 illustrates that the τ-value of a balanced game v is a compromise between an upper bound $M(v)$ and lower bound $m(v)$ for the core. Tijs (1981) mentions several classes of TU-games for which these bounds are sharp, e.g. the class of convex games. A value based on sharp bounds for the core is the β-value introduced in Bondareva (1988) and Bondareva and Driessen (1994). For convex games the τ-value and the β-value coincide. Another value for TU-games which is based on lower and upper values is discussed in van Heumen (1984) and Bergantinos and Masso (1994), who use an upper bound for the core proposed by Milnor (1952). Also van den Brink (1989) considers values for games which are based on upper and lower vectors, although these vectors are not necessarily core bounds.
It is worthwhile to note that also the Shapley value can be regarded as a compromise between a lower and an upper vector[1]: For a game $v \in G^N$ the Shapley value of v is the unique efficient outcome on the line segment between 0 and the sum of the marginal vectors $\sum_{\sigma \in \Pi(N)} m^\sigma(v)$ (this is only true if $v(N) \neq 0$). The idea of regarding

[1]This was pointed out by Carles Rafels.

the Shapley value as a compromise value is used in Chapter 5 where we introduce the marginal based compromise value (MC-value) as a generalization of the Shapley value to the class of monotonic NTU-games.

Driessen and Tijs (1983) provided an alternative approach of calculating the τ-value of quasi-balanced games by introducing the gap function.
Let $v \in G^N$. The *gap function* $g^v : 2^N \to \mathbf{R}$ *of* v is defined by

$$g^v(S) := \sum_{i \in S} M_i(v) - v(S) \text{ for all } S \in 2^N.$$

The gap $g^v(S)$ of coalition S is the difference between the sum of the utopia payoffs of the players in S and the worth of coalition S. The *concession vector* $\lambda(v) \in \mathbf{R}^N$ is defined by

$$\lambda_i(v) := \min_{S: i \in S} g^v(S) \text{ for all } i \in N.$$

The significance of g^v and the vector $\lambda(v)$ follows from the next theorem.

Theorem 2.1.4 (Driessen and Tijs (1983))

(i) $\lambda(v) = M(v) - m(v)$ for every $v \in G^N$

(ii) $QB^N = \{v \in G^N \mid g^v(S) \geq 0 \text{ for all } S \in 2^N, \sum_{i \in N} \lambda_i(v) \geq g^v(N)\}$

(iii) If $v \in QB^N$ and $g^v(N) = 0$, then $\tau(v) = M(v)$

(iv) If $v \in QB^N$ and $g^v(N) > 0$, then $\tau(v) = M(v) - g^v(N)(\sum_{i \in N} \lambda_i(v))^{-1}\lambda(v)$.

Using gap functions Driessen and Tijs introduced several interesting classes of quasi-balanced games for which the τ-value is easy to compute. Here we only mention the class of semi-convex games and the class of 1-convex games. For further classes such as k-convex games the reader is referred to Driessen (1988).
A game $v \in QB^N$ is called *semi-convex* (Driessen and Tijs (1985)) if $g^v(\{i\}) = \min_{S: i \in S} g^v(S)$ for all $i \in N$. Note that a game $v \in QB^N$ is semi-convex if and only if $m_i(v) = v(\{i\})$ for all $i \in N$. Hence, for semi-convex games the τ-value can easily be determined. It is easy to show that convex games are semi-convex.
Further, a game $v \in QB^N$ is called *1-convex* if $g^v(N) = \min_{S \subset N} g^v(S)$. For 1-convex games we have the following analogy of a result of Shapley (1971) who showed that for convex games the Shapley value coincides with the barycenter of the core.

Theorem 2.1.5 (Driessen and Tijs (1983))
If $v \in QB^N$ is 1-convex, then the τ-value and the nucleolus of v both coincide with the barycenter of the core.

We conclude this section with two further brief remarks regarding the τ-value.
Driessen and Tijs (1992) extended the τ-value to TU-games with coalition structures. A *coalition structure* in a TU-game is defined to be a partition of the player set. In games with coalition structures it is assumed that instead of the formation of the grand coalition N, the coalitions in the coalition structure will be formed. Hence, in these games payoff vectors should describe possible divisions of the worth of each coalition in the coalition structure between the members of this coalition. Roughly, the idea behind the τ-value for games with coalition structures is simply to compute separately for each coalition in the coalition structure the τ-value in the subgame induced by this coalition.
Bergantinos and Mendez-Naya (1994) implement the τ-value by means of a strategic game in extensive form, i.e., given a TU-game, they define a (non-trivial) strategic game in extensive form which has a unique subgame perfect equilibrium yielding the same payoff vector as the τ-value of the corresponding TU-game.

2.2 Characterizations of the τ-value

In this section we investigate several properties of the τ-value on the class of quasi-balanced games. We start with some basic properties.

Proposition 2.2.1 The τ-value satisfies the following properties on QB^N.

(1) *efficiency:* $\sum_{i \in N} \tau_i(v) = v(N)$ for all $v \in QB^N$.

(2) *individual rationality:* $\tau_i(v) \geq v(\{i\})$ for all $v \in QB^N$ and all $i \in N$.

(3) *the dummy player property:* $\tau_i(v) = v(\{i\})$ for all $v \in QB^N$ and all *dummy players* i in v, i.e., players $i \in N$ such that $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subset N \setminus \{i\}$.

(4) *symmetry:* $\tau_i(v) = \tau_j(v)$ for all $v \in QB^N$ and all *symmetric players* i and j in the game v, i.e., players i and j such that $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subset N \setminus \{i, j\}$.

(5) *covariance*: For all v and all w in QB^N with $w = kv + a$ for some $k \in (0, \infty)$ and $a \in \mathbf{R}^N$ we have $\tau(w) = k\tau(v) + a$. (Here the game $kv + a$ is defined by $(kv + a)(S) := kv(S) + a(S)$ for all $S \in 2^N$).

Shapley (1953) showed that the Shapley value is the unique value on G^N which satisfies the properties (1), (3), (4) and, in addition, *additivity*, which means that the Shapley value of the sum of two games with the same player set is the sum of the Shapley values of the two games. The Shapley value also satisfies (5) but, it does not satisfy (2) on G^N. Other characterizations of the Shapley value can be found in e.g. Young (1985a) and Hart and Mas-Colell (1989).
On the class of games with non-empty imputation set the nucleolus satisfies all properties mentioned above except additivity. Moreover, the nucleolus is *stable*, i.e., the nucleolus of a game belongs to the core, whenever the core is non-empty. The τ-value and the Shapley value in general do not satisfy stability. However, Theorem 2.1.5 illustrates that the τ-value is stable on the class of 1-convex games. Moreover, it is shown in Driessen and Tijs (1985) that the τ-value is stable on QB^N if there are only two or three players in N. Also for semi-convex quasi-balanced games with four players the τ-value belongs to the core. In Driessen and Tijs (1985) necessary and sufficient conditions for stability of the τ-value are given.
Characterizations of the nucleolus are provided by Sobolev (1975), Snijders (1991), and Potters (1991).

The rest of this section is devoted to characterizations of the τ-value. First, Theorem 2.2.2 considers several additional properties of the τ-value.

Theorem 2.2.2 The τ-value satisfies the following properties on QB^N.

(6) *dummy out property:* If $v \in QB^N$ and $D \subset N$ is the set of dummy players in v, then $\tau(v_{|N \setminus D}) = \tau(v)_{N \setminus D}$ (Here, $v_{|N \setminus D}$ denotes the restriction of v to $N \setminus D$).

(7) *complementary monotonicity:* If $v, w \in QB^N$ are such that $v(T) < w(T)$ for some $T \in 2^N$, $T \neq N$, and $v(S) = w(S)$ for all $S \in 2^N$, $S \neq T$, then $\tau_i(v) \geq \tau_i(w)$ for all $i \in N \setminus T$.

(8) *restricted proportionality:* $\tau(v)$ is proportional to $M(v)$ if $m_i(v) = 0$ for all $i \in N$.

(9) *minimal right property:* $\tau(v) = m(v) + \tau(v - m(v))$ for all $v \in QB^N$.

The dummy out property and the complementary monotonicity property for the τ-value are proved in Tijs and Driessen (1986) and Driessen (1985). Complementary monotonicity of the τ-value means that if a game v is changed to a game w by increasing only the worth of one coalition $T \neq N$ then, according to the τ-value, no player outside T does profit from this deviation. The reader can easily verify that also the Shapley value satisfies the complementary monotonicity property. However, the nucleolus fails to have this property. For a detailed survey of monotonicity properties of the Shapley value, the nucleolus, and the τ-value the reader is referred to Driessen (1985), Otten (1990), and Sagonti (1991).
The minimal right property says that if for a game $v \in QB^N$ first the minimal rights are paid to all the players and then the remaining amount $v(N) - \sum_{i \in N} m_i(v)$ is divided according to the τ-value of the (adapted) game $v - m(v)$, this gives the same allocation as the τ-value of the original game v. Note that the minimal right property is implied by covariance.
The restricted proportionality property and the minimal right property of the τ-value are used in Tijs (1987) to provide the following characterization of the τ-value.

Theorem 2.2.3 (Tijs (1987))
The τ-value is the unique value on QB^N which satisfies efficiency, restricted proportionality, and the minimal right property.

Another characterization of the τ-value on QB^N is provided by Calvo et al. (1993). In this characterization three additional properties of the τ-value play a role, namely bounded aspirations, convexity, and restricted linearity. It turns out that together with efficiency and covariance these three properties characterize the τ-value on QB^N. For more details on this characterization the reader is referred to Calvo et al. (1993). The characterizations of Tijs (1987) and Calvo et al. (1993) are characterizations of the τ-value on a fixed player set N. Recently, Driessen provided a characterization of the τ-value on a set of games with a variable number of players, using a consistency principle. For more details on this characterization the reader is referred to Driessen (1993).

2.3 Bargaining problems

Also in bargaining theory a well-known compromise solution appears, i.e., the Raiffa-Kalai-Smorodinsky solution, or shortly, RKS-solution (Raiffa (1953), Kalai and Smorodinsky (1975)). This solution concept plays a central role in this section.

We start with some basic definitions.

A *bargaining problem* for N is a pair (C, d) where $\emptyset \neq C \subset \mathbf{R}^N$, and $d \in \mathbf{R}^N$ are such that

(i) C is closed, convex, and *comprehensive*, i.e., if $x \in C$ and $y \in \mathbf{R}^N$ are such that $y \leq x$, then $y \in C$

(ii) $C_d := \{x \in C \mid x \geq d\}$ is bounded

(iii) there is an $x^0 \in C$ with $x^0 > d$.

By BP^N we denote the class of all bargaining problems for N.

The interpretation of a bargaining problem (C, d) is as follows. The players in N try to reach an agreement on some outcome x in the *feasible set* C, yielding utility x_i for player $i \in N$. If the players in N do not reach an agreement, then the *disagreement outcome* d results with utility d_i for player $i \in N$. Condition (iii) implies that the players will have an incentive to reach an agreement. The problem of interest is on which outcome should the players in N agree? Many solutions to solve this problem have been proposed.

A *bargaining solution on* BP^N is a map $f : BP^N \to \mathbf{R}^N$ such that $f(C, d) \in C$ for all $(C, d) \in BP^N$. A well-known bargaining solution is the Nash bargaining solution introduced by Nash (1950). The *Nash (bargaining) solution* of a bargaining problem $(C, d) \in BP^N$, denoted $N(C, d)$, is the unique point in C_d where the function

$$x \mapsto \prod_{i \in N} (x_i - d_i)$$

is maximal.

An alternative bargaining solution, first proposed by Raiffa (1953), and characterized by Kalai and Smorodinsky (1975), is the RKS-solution. This solution is a feasible compromise between the disagreement point and a utopia point.

Let $(C, d) \in BP^N$ be a bargaining problem and let $i \in N$. The *utopia point* for player i is the point

$$u_i(C, d) := \max\{x_i \mid x \in C_d\}.$$

Hence, $u_i(C, d)$ is the maximal utility player i can obtain in the set C_d of 'true' bargaining points (a point in $C \setminus C_d$ is not credible as an outcome of the bargaining problem, because it can be improved upon by some players who will prefer the disagreement outcome). The point $u(C,d) := (u_i(C,d))_{i\in N}$ is called the ***utopia point of*** (C,d). The ***RKS-solution*** of (C,d), denoted by $RKS(C,d)$, is defined as the unique weak Pareto optimal point in C lying on the line through d and $u(C,d)$. Here, a point $x \in C$ is called ***weak Pareto optimal in*** C if there does not exist a point $y \in C$ with $y > x$. The set of all weak Pareto optimal points in C is denoted by $WPar(C)$.

Example 2.3.1 Let $N := \{1,2\}$. Consider the bargaining problem (C,d) on N given by $d := (0,0)$ and $C := \{x = (x_1, x_2) \in \mathbf{R}^N \mid x_2 \leq 4,\ 2x_1 + x_2 \leq 8\}$. See Figure 2.1. From Figure 2.1 it follows that $N(C,d) = (2,4)$ and $u(C,d) = (4,4)$. Hence, $RKS(C,d) = (8/3, 8/3)$.

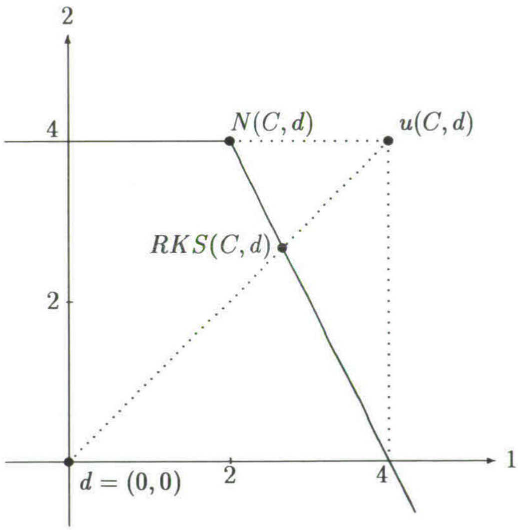

Figure 2.1.

Now we introduce some interesting properties for bargaining solutions.

(i) A bargaining solution $f : BP^N \to \mathbf{R}^N$ is called *Pareto optimal* if for all $(C,d) \in BP^N$ we have $f(C,d) \in Par(C) := \{x \in C \mid y \in C,\ y \geq x \text{ implies } y = x\}$.

(ii) A bargaining solution $f : BP^N \to \mathbf{R}^N$ is called *weakly Pareto optimal* if for all $(C,d) \in BP^N$ we have $f(C,d) \in WPar(C)$.

(iii) A bargaining solution $f : BP^N \to \mathbf{R}^N$ is called *symmetric* if for all $(C,d) \in BP^N$ with $d_i = d_j$ for all $i,j \in N$ and C such that $x \in C$ implies $\sigma(x) \in C$ for all permutations $\sigma \in \Pi(N)$, we have $f_i(C,d) = f_j(C,d)$ for all $i,j \in N$.

(iv) A bargaining solution $f : BP^N \to \mathbf{R}^N$ has the *covariance (with affine transformations) property* if for all $(C,d) \in BP^N$ and all affine functions $A : \mathbf{R}^N \to \mathbf{R}^N$ with $A(x) = \alpha * x + \beta$, $x \in \mathbf{R}^N$, for some $\alpha \in \mathbf{R}^N_{++}$ and $\beta \in \mathbf{R}^N$, we have $f(A(C), A(d)) = A(f(C,d))$.

(v) A bargaining solution $f : BP^N \to \mathbf{R}^N$ satisfies *independence of irrelevant alternatives (IIA)* if for all (C,d), $(D,d) \in BP^N$ with $C \subset D$ and $f(D,d) \in C$ we have $f(C,d) = f(D,d)$.

(vi) A bargaining solution $f : BP^N \to \mathbf{R}^N$ has the *(restricted) monotonicity property* if for all $(C,d), (D,d) \in BP^N$ with $C \subset D$ and $u(C,d) = u(D,d)$ we have $f(C,d) \leq f(D,d)$.

Nash (1950) proved that, in case $|N| = 2$, the Nash solution is the unique bargaining solution which satisfies the properties (i) (or (ii)), (iii), (iv), and (v). Later, this result was extended to bargaining problems with more than two players.
The main property in this characterization is the IIA property, to which much criticism was raised (see, for example Luce and Raiffa (1957), and Kalai and Smorodinsky (1975)). As an alternative for IIA, Kalai and Smorodinsky (1975) suggested a monotonicity property, which is very much related to the (restricted) monotonicity property (cf. Peters (1992)). The replacement of IIA by the (restricted) monotonicity property leads to the following result.

Theorem 2.3.2 (cf. Kalai and Smorodinsky (1975))
The RKS-solution is the unique bargaining solution on the class of two player bargaining problems which satisfies the properties (i) (or (ii)), (iii), (iv), and (vi).

An alternative characterization of the RKS-solution using a consistency principle was obtained by Peters et al. (1991, 1994). Moreover, in this paper the RKS-solution is implemented by the unique subgame perfect equilibrium of a non-cooperative game in extensive form. Another non-cooperative game leading to the RKS-solution was developed earlier in Moulin (1984).

A variation on the (classical) bargaining problem we consider in this section is the bargaining with claims model introduced by Chun and Thomson (1992). In this problem not only a feasible set and a disagreement outcome play a role, but there is also an (infeasible) point which represents the claims or expectations of the players. Also for this type of problems several solutions have been proposed. One of these solutions is the adjusted proportional solution (Herrero (1993)), which assigns to each problem a compromise between a lower value based on an endogenous reference point and the point of claims. The idea behind this solution was derived from Curiel et al. (1987) where the adjusted proportional solution for bankruptcy problems is introduced (cf. Section 2.5).

In the next section we will see that, by weakening some of the properties which characterize the RKS-solution for two player bargaining problems, one can obtain an extension of Theorem 2.3.2 to a large class of NTU-games.

2.4 NTU-games

In this section we consider the more general class of NTU-games which comprises the class of TU-games and the class of bargaining problems. NTU-games were introduced by Aumann and Peleg (1960) and, compared to the above mentioned classes of games the theory on NTU-games is much less developed.

A *non-transferable utility game* or *NTU-game* is a pair (N, V), where N is a finite set of players and V is a map assigning to each coalition $S \in 2^N \setminus \{\emptyset\}$ a subset $V(S)$ of $\mathbb{R}^S$ of *attainable payoff vectors*. We assume that for each $i \in N$ there exists a real number $v(i)$ such that $V(\{i\}) = \{x \in \mathbb{R} \mid x \leq v(i)\}$. Further, we assume that for each $S \in 2^N \setminus \{\emptyset\}$ the following properties hold.

(i) $V(S)$ is a non-empty, closed, and comprehensive subset of $\mathbb{R}^S$

(ii) $V(S) \cap \{x \in \mathbb{R}^S \mid x_i \geq v(i) \text{ for all } i \in S\}$ is bounded.

Note that we do not require the sets $V(S)$ to be convex. So this allows for utility functions which are not necessarily of the von Neumann-Morgenstern type (cf.Kalai and Samet (1985)).
Similar to TU-games we will often identify an NTU-game (N, V) with V.

The next two examples illustrate that the class of NTU-games comprises the class of TU-games and the class of bargaining problems.

Example 2.4.1 Let (N, v) be a TU-game. (N, v) gives rise to an NTU-game (N, V), where for each $S \in 2^N \setminus \{\emptyset\}$

$$V(S) := \{x \in \mathbf{R}^S \mid x(S) \leq v(S)\}.$$

The game V defined above is called *the NTU-game corresponding to* v.

Example 2.4.2 Each bargaining problem (C, d) for N corresponds to an NTU-game (N, V), where

$$\begin{aligned} V(N) &:= C \\ V(S) &:= \{x \in \mathbf{R}^S \mid x_i \leq d_i \ \forall i \in S\} \ \text{ for all } S \in 2^N \setminus \{\emptyset, N\}. \end{aligned}$$

V is called *the NTU-game corresponding to* (C, d).

Similar to the TU-case solution concepts (values, rules) have been defined for NTU-games. Prominent solution concepts for NTU-games are the Harsanyi value (Harsanyi (1963)) and the Shapley NTU-value (Shapley (1969)). These solution concepts are characterized in Hart (1985) and Aumann (1985a), respectively. Other well-known solutions for NTU-games are the egalitarian solution (Kalai and Samet (1985)) and the consistent Shapley value (Maschler and Owen (1989), (1992)). These four solution concepts are extensions of the Shapley value for TU-games to the class of NTU-games. In this section we will not discuss the above mentioned solution concepts in detail, but instead we concentrate on an extension of the τ-value to (a subclass of) NTU-games, which is called the compromise value for NTU-games (when there is no confusion about the class of games under consideration, we will simply talk about the compromise value and omit the addition 'for NTU-games'). The compromise value has been introduced in Borm et al. (1992) and, similar to the τ-value for quasi-balanced TU-games, it is based on upper and lower bounds for the core of an NTU-game.

Let (N, V) be an NTU-game. For each $S \in 2^N \setminus \{\emptyset\}$, let

$$dom(V(S)) := \{x \in \mathbf{R}^S \mid x < y \text{ for some } y \in V(S)\}.$$

The elements of $dom(V(S))$ are elements which are dominated by coalition S.
The *core of* (N, V), denoted $C(V)$, consists of all payoff vectors attainable for the grand coalition N which are not dominated by any coalition S, i.e.,

$$C(V) := \{x \in V(N) \mid x_S \notin dom(V(S)) \text{ for all } S \in 2^N \setminus \{\emptyset\}\}.$$

Let $i \in N$. The *utopia payoff* for player i, $K_i(V)$, is defined by

$$K_i(V) := \sup\{t \in \mathbf{R} \mid \exists_{a \in \mathbf{R}^{N\setminus\{i\}}} : (a,t) \in V(N), a \notin dom(V(N \setminus \{i\})),$$

$$a \geq (v(j))_{j \in N\setminus\{i\}}\}.$$

By assumption (ii) in the definition of an NTU-game it follows that $K_i(V) < \infty$. However, it might happen that $K_i(V) = -\infty$. We restrict ourselves to NTU-games (N,V) for which $K_i(V) \in \mathbf{R}$ for all $i \in N$. The vector $K(V) := (K_i(V))_{i\in N}$ is called the *upper value* of V.
Let $i \in N$ and let $S \in 2^N$ with $i \in S$. The *remainder* of $i \in S$ is given by

$$\rho^V(S,i) := \sup\{t \in \mathbf{R} \mid \exists_{a \in \mathbf{R}^{S\setminus\{i\}}} : (a,t) \in V(S), a > K_{S\setminus\{i\}}(V)\}.$$

The *minimal right* of player i is denoted by

$$k_i(V) := \max_{S: i \in S} \rho^V(S,i),$$

and the vector $k(V) := (k_i(V))_{i\in N}$ is called the *lower value* for V. Again, we restrict ourselves to NTU-games (N,V) for which $k(V) \in \mathbf{R}^N$.

Analogously to Theorem 2.1.2 we have the next result.

Theorem 2.4.3 (Borm et al. (1992))
If (N,V) is an NTU-game with $C(V) \neq \emptyset$, then

$$k(V) \leq x \leq K(V) \text{ for all } x \in C(V).$$

Moreover, we have the following theorem.

Theorem 2.4.4 (Borm et al. (1992))

(i) Let (N,v) be a TU-game with $v(N \setminus \{i\}) \geq \sum_{j\in N\setminus\{i\}} v(\{j\})$ for all $i \in N$ and let (N,V) be the corresponding NTU-game. Then $K(V) = M(v)$ and $k(V) = m(v)$.

(ii) Let (C,d) be a bargaining problem for N, and let (N,V) be the corresponding NTU-game. Then $K(V) = u(C,d)$ and $k(V) = d$.

The compromise value is defined on the class of compromise admissible NTU-games. An NTU-game (N, V) is called *compromise admissible* if

$$k(V) \leq K(V),\ k(V) \in V(N),\ \text{and}\ K(V) \notin dom(V(N)).$$

By C^N we denote the class of all compromise admissible NTU-games with player set N. From Theorem 2.4.3 it immediately follows that $V \in C^N$ if $C(V) \neq \emptyset$. Furthermore, from Theorem 2.4.4 it follows that NTU-games corresponding to bargaining problems are compromise admissible, and that for a quasi-balanced TU-game (N, v) with $v(N \setminus \{i\}) \geq \sum_{j \in N \setminus \{i\}} v(\{j\})$ for all $i \in N$, the corresponding NTU-game is compromise admissible.

For a compromise admissible NTU-game (N, V) the *compromise value* $T(V)$ is defined as the unique vector on the line segment between $k(V)$ and $K(V)$ which belongs to $V(N)$ and is nearest to the utopia value $K(V)$, i.e.,

$$T(V) := k(V) + \alpha_V(K(V) - k(V)),$$

where

$$\alpha_V := \max\{\alpha \in [0, 1] \mid k(V) + \alpha(K(V) - k(V)) \in V(N)\}.$$

The following corollary immediately follows from Theorem 2.4.4.

Corollary 2.4.5 ((Borm et al. (1992))

(i) If v is a quasi-balanced TU-game satisfying $v(N \setminus \{i\}) \geq \sum_{j \in N \setminus \{i\}} v(\{j\})$ for all $i \in N$, and (N, V) is the corresponding NTU-game, then $\tau(v) = T(V)$.

(ii) If (C, d) is a bargaining problem for N, and (N, V) is the corresponding NTU-game, then $RKS(C, d) = T(V)$.

So the compromise value definitionally extends the τ-value and the RKS-solution to NTU-games. As Theorem 2.4.6 and Theorem 2.4.7 (or 2.4.9) below show, both the characterization of the τ-value by Tijs (1987) (Theorem 2.2.3) and the characterization of the two player RKS-solution by Kalai and Smorodinsky (1975) (Theorem 2.3.2) can be extended in order to provide characterizations of the compromise value. For this we introduce the following properties of values for NTU-games which are extensions of properties for values for TU-games and solutions for bargaining problems. Let $f : C^N \to \mathbf{R}^N$ be a value on the set of compromise admissible games with player set N. Then

(i) f satisfies *weak Pareto optimality* if $f(V) \in V(N) \setminus dom(V(N))$ for all $V \in C^N$

(ii) f satisfies *covariance* if for all $V \in C^N$, all $\alpha \in \mathbf{R}^N_{++}$ and all $\beta \in \mathbf{R}^N$ we have $f(\alpha * V + \beta) = \alpha * f(V) + \beta$. (The game $\alpha * V + \beta$ is defined by $(\alpha * V + \beta)(S) := \alpha_S * V(S) + \{\beta_S\}$ for all $S \in 2^N \setminus \{\emptyset\}$.)

(iiii) f is *monotonic* if for all $V, W \in C^N$ with $k(V) = k(W), K(V) = K(W)$ and $V(N) \subset W(N)$ we have $f(V) \leq f(W)$

(iv) f is called *bounds-symmetric* if for all $V \in C^N$ and all $i, j \in N$ with $k_i(V) = k_j(V), K_i(V) = K_j(V)$, we have $f_i(V) = f_j(V)$

(v) f satisfies *symmetry* if $f_i(V) = f_j(V)$ for all $V \in A^N$ and all $i, j \in N$ which are symmetric in V. Here, players $i, j \in N$ are called *symmetric* in V if

(1) for all $S \subset N \setminus \{i, j\}$, all $x \in V(S \cup \{i\})$ it holds that $y \in V(S \cup \{j\})$, where $y \in \mathbf{R}^{S \cup \{j\}}$ is defined by $y_j = x_i$ and $y_S = x_S$,

(2) for all $S \subset N$, $i, j \in S$, and all $x \in V(S)$, we have $y \in V(S)$, where $y \in \mathbf{R}^S$ is defined by $y_i = x_j$, $y_j = x_i$ and $y_{S \setminus \{i,j\}} = x_{S \setminus \{i,j\}}$.

(vi) f satisfies *restricted proportionality* if $f(V)$ is proportional to $K(V)$ for all $V \in C^N$ with $k(V) = 0$

(vii) f satisfies the *minimum right property* if $f(V) = k(V) + f(V - k(V))$ for all $V \in C^N$

Clearly, the compromise value satisfies all properties mentioned above.
It turns out that (i), (vi) and (vii) characterize the compromise value on the class of all compromise admissible games (N, V) for which the boundary of the set $V^*(N) := \{x \in V(N) \mid x \geq k(V)\}$ contains no segments parallel to a coordinate hyperplane, i.e.,

$V^*(N)$ is *non-level*: $x, y \in V^*(N) \setminus dom(V(N))$ and $x \geq y$, implies $x = y$.(2.1)

Let $\overline{C}^N$ denote the class of all compromise admissible games satisfying (2.1). The non-levelness condition is a standard condition often used in characterizations of solution concepts for NTU-games (cf. Aumann (1985a)). Analogously to Theorem 2.2.3, we now have the following result.

Theorem 2.4.6 (Borm et al. (1992))
The compromise value is the unique value on $\overline{C}^N$ which satisfies weak Pareto optimality, restricted proportionality, and the minimum right property.

Borm et al. (1992) show that the properties (i)-(iv) characterize the compromise value on the smaller subclass $\tilde{C}^N \subset \overline{C}^N$ of compromise admissible games (N,V) satisfying

$$k(V) < K(V) \tag{2.2}$$

$$(k_{N\setminus\{i\}}(V), K_i(V)) \in V(N) \text{ for all } i \in N \tag{2.3}$$

$$V(N) \text{ is convex.} \tag{2.4}$$

Theorem 2.4.7 (Borm et al. (1992))
The compromise value is the unique value on $\tilde{C}^N$ which satisfies weak Pareto optimality, bounds-symmetry, monotonicity, and covariance.

It turns out that this theorem can be strengthened, since in this characterization the monotonicity property is superfluous. As a consequence of this observation we obtain a new characterization of the compromise value.

Theorem 2.4.8 (Otten, Borm and Tijs (1994))
The compromise value is the unique value on $\tilde{C}^N$ which satisfies

(i) weak Pareto optimality,

(ii) bounds-symmetry,

(iii) covariance.

Proof. We already know that the compromise value satisfies the three properties on $\tilde{C}^N$. Let $f : \tilde{C}^N \to \mathbf{R}^N$ satisfy the three properties, and let $V \in \tilde{C}^N$. We show that $f(V) = T(V)$.
Let $V' := V - k(V)$. Clearly, $V' \in \tilde{C}^N$ and $k(V') = 0$. Moreover by (2.2), $K(V') = K(V) - k(V) > 0$. Define $\lambda \in \mathbf{R}^N$ by $\lambda_i := (K_i(V'))^{-1}$ for all $i \in N$. Then $\lambda > 0$. Let $W := \lambda * V'$. Then $W \in \tilde{C}^N$, $k(W) = \lambda * k(V') = 0$, and $K(W) = \lambda * K(V') = e^N$. Bounds-symmetry of f and T implies $f_i(W) = f_j(W)$ for all $i,j \in N$ and $T_i(W) = T_j(W)$ for all $i,j \in N$. From weak Pareto optimality of f and T it follows

that $f(W) = T(W)$. Since $V = K(V') * W + k(V)$, covariance of f and T implies $f(V) = T(V)$. □

Theorem 2.4.8 is similar to one of the characterizations of the MC-value discussed in Chapter 5 (Theorem 5.4.2). Note that in the proof of Theorem 2.4.8 we did not use the conditions (2.1), (2.3), and (2.4). So Theorem 2.4.8 can be adjusted to a characterization on the larger class of compromise admissible NTU-games satisfying only (2.2).

The original proof of Theorem 2.4.7 provided by Borm et al. (1992) actually shows the following characterization of the compromise value on $\widetilde{C}^N$ in which the bounds-symmetry property is replaced by the weaker and more natural symmetry property.

Theorem 2.4.9 The compromise value is the unique value on $\widetilde{C}^N$ which satisfies

(i) weak Pareto optimality,

(ii) symmetry,

(iii) monotonicity,

(iv) covariance.

The following four examples show that in Theorem 2.4.9 the properties are independent.

(I) The value $f : \widetilde{C}^N \to \mathbf{R}^N$ defined by $f(V) := K(V)$ for all $V \in \widetilde{C}^N$ satisfies symmetry, monotonicity and covariance, but not Pareto optimality.

(II) In Section 4.2 we introduce values corresponding to monotonic curves. These solution concepts satisfy Pareto optimality, monotonicity, and covariance, but not necessarily symmetry.

(III) The MC-value defined in Chapter 5 satisfies Pareto optimality, symmetry, and covariance, but not monotonicity.

(IV) Let $f : \widetilde{C}^N \to \mathbf{R}^N$ be defined as follows: If $V \in \widetilde{C}^N$ is such that $K(V) > 0$ and $V(N) \cap \mathbf{R}^N_{++} \neq \emptyset$, then $f(V)$ is the unique element of $V(N) \backslash wdom(V(N))$ which belongs to the line segment between 0 and $K(V)$. Otherwise, $f(V) := T(V)$. The value f satisfies Pareto optimality, symmetry, and monotonicity, but not covariance.

Besides the properties mentioned above, the compromise value also satisfies other standard properties, such as individual rationality and the dummy player property. Additional properties of the compromise value such as the dummy out property and a complementary monotonicity property which is slightly different from the complementary monotonicity property of the τ-value are studied in Otten (1990). An extension of the compromise value to NTU-games with coalition structures can also be found in Otten (1990).

Borm et al. (1992) provided another extension of the τ-value to NTU-games, namely the NTU τ-value. The NTU τ-value is based on the same ideas as the Shapley NTU-value (Shapley (1969)). Given an NTU-game, Shapley considered so-called λ-transfer TU-games associated with this game. The Shapley NTU-value is obtained from the Shapley values of these TU-games. Similarly, the NTU τ-value is obtained from the τ-values of quasi-balanced λ-transfer games.
Let (N, V) be an NTU-game and let $\lambda \in \Delta_N := \{x \in \mathbf{R}^N \mid x \geq 0, \sum_{i \in N} x_i = 1\}$. The vector λ is called *V-feasible* if for all $S \in 2^N \setminus \{\emptyset\}$

$$v_\lambda(S) := \sup\{\sum_{i \in S} \lambda_i x_i \mid x \in V(S)\} < \infty.$$

So, a V-feasible λ generates a TU-game (N, v_λ). This TU-game is called a *λ-transfer game* corresponding to (N, V). If for all V-feasible $\lambda \in \Delta_N$ the corresponding λ-transfer games are quasi-balanced, the game (N, V) is called *τ-admissible.* For a τ-admissible NTU-game (N, V) the *NTU τ-value*, denoted by $\tau(V)$, is defined by

$$\tau(V) := \{x \in V(N) \mid \text{ there is a } V\text{-feasible } \lambda \in \Delta_N \text{ such that } \tau(v_\lambda) = \lambda * x\}.$$

Note that the NTU τ-value of an NTU-game not necessarily consists of one point, so the name *value* is rather misleading here (the same problem occurs with the Shapley NTU-value which is not single-valued either). The NTU τ-value can even be empty for τ-admissible games. In Borm et al. (1992) a class of τ-admissible NTU-games is given for which the NTU τ-value is non-empty.
If (N, v) is a quasi-balanced TU-game, then the corresponding NTU-game is τ-admissible and the NTU τ-value of this NTU-game coincides with the τ-value of v. Moreover, for two player bargaining situations the NTU τ-value coincides with the Nash bargaining solution.
An extension of the NTU τ-value to NTU-games with coalition structures can be found in Otten (1990).

2.5 Applications

This section discusses several applications of cooperative game theory to problems in economics and operations research. We pay special attention to compromise values and compare the outcomes prescribed by compromise values to the outcomes of other game theoretic solution concepts. It should be mentioned that the list of applications we present in this section is certainly not complete. An important application of cooperative game theory is the analysis of cost allocation problems. The study of cost allocation problems in a game theoretical framework is the subject of Chapter 3 where we, in particular, investigate the so-called ACA-method, a compromise value that was proposed by engineers of the Tennessee Valley Authority (TVA).

It turns out that for some classes of games we discuss in this section the τ-value coincides with the Shapley value or the nucleolus. This raises the interesting question to find necessary and sufficient conditions for coincidence of the τ-value, Shapley value, and the nucleolus.

Two classes of so-called combinatorial optimization games (weighted graph games and sequencing games) are discussed. For these classes the τ-value is easy to compute. However, for other combinatorial optimization games such as flow games (Kalai and Zemel (1982), Curiel et al. (1989)), traveling salesman games (Fishburn and Pollak (1983), Tamir (1989), Potters et al. (1992)), and minimum cost spanning tree games (Granot and Huberman (1981)) no explicit formulas for the τ-value are known. Although for a special type of minimum cost spanning tree games, called minimal chain games, the τ-value yields attractive outcomes (cf. Aarts (1994)), the above-mentioned classes of combinatorial optimization games are not discussed in this section. A recent survey on combinatorial optimization games is provided by Tijs (1992).

Bankruptcy problems

A *bankruptcy problem* is a pair $(E, d) \in \mathbf{R} \times \mathbf{R}^N$, where $d_i \geq 0$ for all $i \in N$ and $0 \leq E \leq \sum_{i \in N} d_i$. Here, E is the estate which has to be divided among the claimants in N, and d_i is the claim of claimant $i \in N$. Several allocation rules for bankruptcy problems have been proposed. An *allocation rule* is a function f which assigns to every bankruptcy problem (E, d) a vector $f(E, d) \in \mathbf{R}^N$ such that

(i) $0 \leq f_i(E, d) \leq d_i$ for all $i \in N$

(ii) $\sum_{i \in N} f_i(E, d) = E$.

Some examples of allocation rules are the proportional rule, which divides the estate proportional to the claims of the creditors and the constrained equal award rule. We concentrate on the adjusted proportional rule as introduced and characterized by Curiel et al. (1987).

The *adjusted proportional rule*, or *AP-rule*, starts by giving each claimant $i \in N$ his *minimal right* m_i, which is the maximum of zero and the amount not claimed by the other claimants, i.e., $m_i := \max\{E - \sum_{j \in N\setminus\{i\}} d_j, 0\}$. Next, the amount E' of the estate which is left, i.e., $E' := E - \sum_{i \in N} m_i$, has to be divided. Because each claimant already received a part of his claim the claims are lowered. The claim of claimant $i \in N$ on E' becomes $d'_i := \min\{d_i - m_i, E'\}$ (claims higher than E' are considered irrational). Now the remaining estate E' is divided proportionally to the new claims $(d'_i)_{i \in N}$.

Example 2.5.1 Consider the bankruptcy problem (E, d) with $E = 400$, and $d = (100, 200, 300)$. To determine $AP(E, d)$ we first compute the minimal rights of the players:

$$m_1 = \max\{400 - 200 - 300, 0\} = 0,$$
$$m_2 = \max\{400 - 100 - 300, 0\} = 0,$$
$$m_3 = \max\{400 - 100 - 200, 0\} = 100.$$

The remaining estate $E' = E - \sum_{i \in N} m_i = 300$ and the vector of new claims becomes $d' = (100, 200, 200)$. Hence,

$$AP(E, d) = (0, 0, 100) + \frac{300}{500}(100, 200, 200) = (60, 120, 220).$$

The AP-rule satisfies several nice properties. Some of them are listed below.

(i) The AP-rule satisfies the *minimal right property*, which states that it makes no difference whether the rule is directly applied to a given bankruptcy situation, or that first the minimal rights are allocated to the players and then the AP-rule is applied on the remaining estate and the adjusted claims.

(ii) The AP-rule is *symmetric*, which means that if two claimants have the same claims, they are also allocated the same part of the estate.

(iii) The AP-rule satisfies the *truncated claim property*, which means that, given a bankruptcy problem, it does not matter for the final allocation whether all claims higher than the estate are replaced by claims equal to the estate.

(iv) The AP-rule satisfies the *additivity of claims property.* This property states that, given a bankruptcy problem (E, d) satisfying $m_i = 0$ for all $i \in N$, if one of the claimants dies leaving behind parts of his claim to different heirs, which become new claimants, this does not affect the allocation to the other claimants.

It turns out that the four properties listed above are sufficient to characterize the AP-rule.

Theorem 2.5.2 (Curiel et al. (1987))
The AP-rule is the unique allocation rule for bankruptcy problems which satisfies the minimal right property, symmetry, the truncated claim property, and additivity of claims.

For a bankruptcy problem $(E, d) \in \mathbf{R} \times \mathbf{R}^N$, the corresponding *bankruptcy game* $(N, v_{E,d})$ is defined by (cf. O'Neill (1982))

$$v_{E,d}(S) := \max\{E - \sum_{i \in N \setminus S} d_i, 0\} \quad \text{for all } S \in 2^N.$$

In Curiel et al. (1987) it is shown that bankruptcy games are convex games, and hence, the τ-value can easily be computed.

Example 2.5.3 Consider the bankruptcy problem (E, d) of Example 2.5.1. The corresponding bankruptcy game $v := v_{E,d}$ is given by

$$v(\{1\}) = v(\{2\}) = 0,\ v(\{3\}) = v(\{1,2\}) = 100,$$
$$v(\{1,3\}) = 200,\ v(\{2,3\}) = 300,\ v(N) = 400.$$

If we compute the τ-value of this TU-game, we find that $\tau(v) = (60, 120, 220) \in C(v)$. Hence, the τ-value of this bankruptcy game belongs to the core of the game and it coincides with the AP-solution of the bankruptcy problem. This is no coincidence as is shown in the next theorem.

Theorem 2.5.4 (Curiel et al. (1987))
Let (E, d) be a bankruptcy problem and let $(N, v_{E,d})$ be the corresponding bankruptcy game. Then

(i) $AP(E, d) = \tau(v_{E,d})$

(ii) $\tau(v_{E,d}) \in C(v_{E,d})$.

In O'Neill (1982) and Aumann and Maschler (1985) the Shapley value and the nucleolus for bankruptcy games are studied.

An alternative game theoretic approach to bankruptcy problems is introduced by Dagan and Volij (1993). Given a bankruptcy problem (E,d), one can construct a bargaining problem $(C_{(E,d)}, b_{(E,d)})$ in the following way. A natural choice for the set $C_{(E,d)}$ of feasible outcomes is to define

$$C_{(E,d)} := \{x \in \mathbf{R}^N \mid x \leq d, \sum_{i\in N} x_i \leq E\}.$$

Dagan and Volij (1993) proposed two possible alternatives for the disagreement outcome, namely $b_{(E,d)} := 0$ and $b_{(E,d)} := m(E,d)$, where $m(E,d)$ denotes the vector consisting of the minimal rights of the players. In case $b_{(E,d)} = m(E,d)$ we have the following theorem.

Theorem 2.5.5 (Dagan and Volij (1993))
Let (E,d) be a bankruptcy problem and let $(C_{(E,d)}, m(E,d))$ be the corresponding bargaining problem. Then

$$RKS(C_{(E,d)}, m(E,d)) = AP(E,d) = \tau(v_{E,d}).$$

Exchange markets

Many economic situations can be modelled using cooperative game theory. Sometimes it is more natural to use NTU-games than to use TU-games. This is the case for example if one wants to model exchange markets as cooperative games.
An *exchange market* $\mathcal{E}$ is a tuple $< N,\ \mathbf{R}^m_+,\ (f^i)_{i\in N},\ (u_i)_{i\in N} >$. Here, N is the set of agents, $\mathbf{R}^m_+$ is the commodity space, $f^i \in \mathbf{R}^m_+$ is the initial commodity bundle of agent $i \in N$, and $u_i : \mathbf{R}^m_+ \to \mathbf{R}$ is the utility function of agent $i \in N$. An *admissible reallocation* of coalition S is a collection of commodity bundles $(x^i)_{i\in S}$ such that $x^i \in \mathbf{R}^m_+$ for each $i \in S$ and $\sum_{i\in S} x^i = \sum_{i\in S} f^i$. The set of admissible reallocations of coalition S is denoted by $A(S)$.
An exchange market $\mathcal{E}$ gives rise to an NTU-game $(N, V_{\mathcal{E}})$ as follows. For each $S \in 2^N \setminus \{\emptyset\}$ define

$$V_{\mathcal{E}}(S) := \{t \in \mathbf{R}^S \mid \exists_{(x^i)_{i\in S}\in A(S)}[u_i(x^i) \geq t_i \text{ for all } i \in S]\}.$$

The following well-known example of Shafer (1980) has led to an interesting debate on the interpretation of the Shapley NTU-value (cf. Aumann (1985b, 1986), Roth (1986)).

Example 2.5.6 Consider the following exchange market $\mathcal{E}$ with three agents and two commodities. The initial commodity bundles and the utility functions of the agents 1, 2, and 3 are given by

$$f^1 = (1-\epsilon, 0),\ f^2 = (0, 1-\epsilon),\ f^3 = (\epsilon, \epsilon),$$

$$u_1(x_1, x_2) = u_2(x_1, x_2) = \min\{x_1, x_2\}, \text{ and } u_3(x_1, x_2) = \tfrac{1}{2}(x_1 + x_2)$$

for all $(x_1, x_2) \in \mathbf{R}^2_+$, where $0 \leq \epsilon < \frac{1}{5}$.
The corresponding NTU-game $(N, V_{\mathcal{E}})$ is given by

$$\begin{aligned}
&V_{\mathcal{E}}(\{i\}) = \{t \in \mathbf{R} \mid t \leq 0\},\ i = 1,2\\
&V_{\mathcal{E}}(\{3\}) = \{t \in \mathbf{R} \mid t \leq \epsilon\},\\
&V_{\mathcal{E}}(\{1,2\}) = \{(t_1, t_2) \in \mathbf{R}^{\{1,2\}} \mid t_1 + t_2 \leq 1-\epsilon,\ t_1 \leq 1-\epsilon,\ t_2 \leq 1-\epsilon\},\\
&V_{\mathcal{E}}(\{1,3\}) = \{(t_1, t_3) \in \mathbf{R}^{\{1,3\}} \mid t_1 + t_3 \leq \tfrac{1}{2} + \tfrac{1}{2}\epsilon,\ t_1 \leq \epsilon,\ t_3 \leq \tfrac{1}{2} + \tfrac{1}{2}\epsilon\},\\
&V_{\mathcal{E}}(\{2,3\}) = \{(t_2, t_3) \in \mathbf{R}^{\{2,3\}} \mid t_2 + t_3 \leq \tfrac{1}{2} + \tfrac{1}{2}\epsilon,\ t_2 \leq \epsilon,\ t_3 \leq \tfrac{1}{2} + \tfrac{1}{2}\epsilon\},\\
&V_{\mathcal{E}}(\{1,2,3\}) = \{(t_1, t_2, t_3) \in \mathbf{R}^N \mid t_1 + t_2 + t_3 \leq 1,\ t_1 \leq 1,\ t_2 \leq 1,\ t_3 \leq 1\}.
\end{aligned}$$

Easy computations yield that in this case the compromise value and the NTU τ-value give the same solution, namely $(\frac{1}{2} - \frac{1}{2}\epsilon, \frac{1}{2} - \frac{1}{2}\epsilon, \epsilon)$.
The Shapley NTU-value yields the outcome $(\frac{5}{12} - \frac{5}{12}\epsilon, \frac{5}{12} - \frac{5}{12}\epsilon, \frac{1}{6} + \frac{5}{6}\epsilon)$. We see that the Shapley NTU-value always assigns a positive payoff to agent 3 of at least $\frac{1}{6}$ even if $\epsilon = 0$. But if $\epsilon = 0$, agents 1 and 2 together can achieve a utility of 1 by forming the subcoalition $\{1,2\}$, leaving 0 for agent 3. This was the reason that Shafer (1980) (and Roth (1986)) claimed that in this case the Shapley NTU-value is not a reasonable outcome. However, using arguments from non-cooperative game theory, Aumann (1985b, 1986) argues that it can be reasonable that player 3 gets a positive utility. Obviously, the compromise value and the NTU τ-value (and also the Harsanyi value) give a utility of 0 to agent 3 if $\epsilon = 0$.

Big boss games

A TU-game (N, v) is called a *big boss game* (with player i as big boss) (cf. Muto et al. (1988)) if the following three conditions hold:

(i) v is *monotonic*, i.e., if $S \subset T \subset N$, then $v(S) \leq v(T)$

(ii) $v(S) = 0$ if $i \notin S$

(iii) $v(N) - v(S) \geq \sum_{j \in N \setminus S} M_j(v)$ if $i \in S$.

Condition (i) implies that $v(S) \geq 0$ for all $S \in 2^N$ and that $M(v) \geq 0$, and (ii) states that player i is very powerful, i.e., coalitions not containing i cannot get anything. Condition (iii), which Muto et al. (1988) call the 'union property', implies that for a coalition without the big boss the marginal contribution to the grand coalition is at least as large as the sum of the marginal contributions of each of its members. It turns out that there are many economic situations which give rise to big boss games. We mention:

(1) bankruptcy problems with one big claimant, i.e., a claimant who claims at least the estate

(2) one-seller/many buyers situations of a certain type

(3) information market games as introduced in Muto et al. (1989).

For these and more applications the reader is referred to Muto et al. (1988). Generalizations of big boss games were studied in Potters et al. (1989) (clan games) and Nagarajan (1992) (games with leading coalitions).

In the next theorem some results for big boss games are collected.

Theorem 2.5.7 (Muto et al. (1988))
Let (N, v) be a big boss game with player i as the big boss. Then

(i) the core of v is a paralellotope, consisting of all the vectors $x \in \mathbf{R}^N$ with $\sum_{i \in N} x_i = v(N)$ and $0 \leq x_j \leq M_j(v)$ for all $j \in N \setminus \{i\}$

(ii) the τ-value and the nucleolus of v both coincide with the center of the core, i.e.,

$$\tau_j(v) = n_j(v) = \begin{cases} v(N) - \frac{1}{2}\sum_{k \in N \setminus \{j\}} M_k(v) & \text{if } j = i \\ \frac{1}{2} M_j(v) & \text{if } j \neq i \end{cases}$$

(iii) for the Shapley value $\Phi(v)$ we have $\Phi_i(v) \leq \tau_i(v)$

(iv) $\Phi(v) = \tau(v) = n(v)$ if and only if v is convex.

Weighted graph games

Brown and Housman (1988) introduced weighted graph games as a class of games where the value of a coalition with more than two players only depends on the values of the two player subcoalitions. Formally, a *weighted graph game* is a TU-game (N, v) where

$$v := \sum_{T:|T|=2} \alpha_T u_T$$

with $\alpha_T \geq 0$ for all $T \in 2^N$, $|T| = 2$. Here, u_T denotes the T-*unanimity game* defined by

$$u_T(S) := \begin{cases} 1 & \text{if } T \subset S \\ 0 & \text{otherwise.} \end{cases}$$

A weighted graph game corresponds to a weighted complete graph in which the players are the vertices and the weight on an edge $\{i,j\} \subset N$, $i \neq j$, is given by $\alpha_{\{i,j\}}$. For a coalition $S \in 2^N$, $v(S)$ can be seen as the sum of the weights on the edges of the subgraph induced by S.
Weighted graph games form a subclass of the class of convex games. The following theorem illustrates that on this subclass the Shapley value, the nucleolus, and the τ-value coincide.

Theorem 2.5.8 (Brown and Housman (1988))
Let (N,v) be a weighted graph game. Then for all $i \in N$

$$\Phi_i(v) = \tau_i(v) = n_i(v) = \frac{1}{2}(\text{the sum of the weights of all edges adjacent to } i).$$

As a corollary of Theorem 2.5.8 it follows that the τ-value and the nucleolus are additive on the cone

$$K_2^N := cone\{u_T \mid T \in 2^N, |T| = 2\}.$$

In van den Nouweland et al. (1994) it is shown that the τ-value is additive on each of the cones $K_l^N := cone\{u_T \mid T \in 2^N, |T| = l\}$ with $2 \leq l \leq |N|$.

Theorem 2.5.9 (van den Nouweland et al. (1994))
Let $(N,v) \in K_l^N$ $(2 \leq l \leq |N|)$. Then $\Phi(v) = \tau(v)$.

It is not difficult to show that the nucleolus does not coincide with the τ-value and the Shapley value on K_l^N if $l > 2$.

Sequencing games

In a *sequencing situation* there is a queue, consisting of n customers waiting to be served at a counter. The original order of the customers is given by a permutation σ of $N := \{1,\ldots,n\}$. In the sequel we assume without loss of generality that $\sigma(i) = i$

for all $i \in N$. For every $i \in N$, $s_i \in \mathbf{R}_+$ denotes the *service time* of i and $c_i : \mathbf{R}_+ \to \mathbf{R}$ is the *cost function* of i ($c_i(t)$ denotes the costs of customer i if his completion time is t). We assume that c_i is increasing and affine, i.e., $c_i(t) = \alpha_i t + \beta_i$ for all $t \in \mathbf{R}_+$, where $\alpha_i > 0$.
Given a sequencing situation one can construct a TU-game in the following way (cf. Curiel et al. (1989)). The set of players is N, and define the sequencing game (N, v) in such a way that the worth of a coalition S is equal to the maximal cost savings the coalition can obtain by rearranging their positions in the queue. Hereby, we allow two customers in S to change positions only if there is no customer outside S standing between them in the original order. The cost savings that two neighbours i and j in the queue can obtain by switching are $g_{ij} := \max\{\alpha_j s_i - \alpha_i s_j, 0\}$. To see this, note that if i and j change positions, then i's completion time increases with s_j (we assume $i < j$). So his extra costs are $\alpha_i s_j$. Analogously, the cost savings of j are $\alpha_j s_i$. So, the joint cost savings are $\alpha_j s_i - \alpha_i s_j$. If this is negative, then it is not favourable for i and j to change positions. Notice that to compute the cost savings one does not have to know the numbers β_i, $i \in N$.

Example 2.5.10 Consider the following sequencing situation, in which there are three customers in the queue, numbered 1, 2, and 3. So, $N = \{1, 2, 3\}$. The service times of the customers are given by $s_1 = 7$, $s_2 = 3$, and $s_3 = 5$. Further, $\alpha_1 = 10$, $\alpha_2 = 20$, and $\alpha_3 = 30$. If customers 1 and 2 change their positions in the queue then the cost savings that can be obtained are $7 \times 20 - 3 \times 10 = 110$, so $g_{12} = 110$. Easy calculations yield that $g_{13} = 160$, $g_{32} = 10$, and $g_{ij} = 0$ otherwise.

A coalition $T \in 2^N$ is called *connected* if for all $i, j \in T$ and all $k \in N$, with $i < k < j$, we have $k \in T$. Curiel et al. (1989) proved that for a connected coalition T the maximal cost savings are

$$v(T) = \sum_{i,j \in T : i < j} g_{ij}.$$

For a non-connected coalition S, we say that $T \subset S$ is a *component* of S if T is connected and $T \cup \{i\}$ is not connected for every $i \in S \setminus T$. The components of S form a partition of S which we denote by $\mathcal{P}(S)$. Now it follows

$$v(S) = \sum_{T \in \mathcal{P}(S)} v(T).$$

In Curiel et al. (1989) it is shown that sequencing games are convex games, and therefore, the τ-value can easily be computed. For player $i \in N$ the utopia payoff $M_i(v)$ is equal to

$$M_i(v) = \sum_{j,k \in N:\, j<k} g_{jk} - \sum_{j,k \in N:\, j<k<i} g_{jk} - \sum_{j,k \in N:\, i<j<k} g_{jk} = \sum_{j,k \in N:\, j \neq k,\, j \leq i \leq k} g_{jk}.$$

This last expression resembles precisely the cost savings that cannot be obtained if player i does not cooperate. Since sequencing games are zero-normalized, i.e., $v(\{i\}) = 0$ for all $i \in N$, it follows that the τ-value is proportional to the upper value.

Example 2.5.11 Consider again the sequencing situation of Example 2.5.10. The corresponding sequencing game (N, v) is defined by

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0,$$
$$v(\{1,2\}) = 110,\ v(\{1,3\}) = v(\{2,3\}) = 0,\ v(N) = 270.$$

The τ-value of v is given by $\tau(v) = \frac{27}{70}(270, 270, 160)$.

Chapter 3

Cost allocation problems

Cost allocation problems occur in many real life situations, where individuals work together in a joint project. In these cases the problem arises of allocating the joint costs to the participants in the project in a "fair" way. Cooperative game theory provides a mathematical tool to analyse this type of problems.

Examples of cost allocation problems studied in a game theoretical context are the setting of fees for common facilities like communication networks, canals, airports, etc. Other examples are the allocation of joint overhead costs of a firm among its different divisions (e.g. Shubik (1962), Jensen (1977), Hamlen et al. (1977)), and the apportioning of costs of multipurpose water projects. (e.g. Ransmeier (1942), Suzuki and Nakayama (1976), Loughlin (1977), Straffin and Heaney (1981), Young et al. (1982)).

Especially the last type of cost allocation problems has a rich history dating back to the 1930's in which the Tennessee Valley Authority (TVA) was established to improve the economic situation in the Tennessee Valley region. Among the projects initiated by the TVA were several multipurpose water projects in the Tennessee River. The problem TVA engineers were confronted with was that the TVA Act of 1933 required that the costs of a project had to be apportioned among the different 'purposes' to be served by the project. The TVA developed several methods to allocate the costs of projects to the purposes. One of these methods is the alternate cost avoided method, or shortly, the ACA-method, which was proposed by Martin Glaeser, an engineer of the TVA in 1938.

A game theoretical basis for the ACA-method was established by Gately (1974). Gately proposed a new solution concept for cooperative games based on a player's "propensity to disrupt" the solution. This solution concept has been further generalized by Fischer and Gately (1975), Littlechild and Vaidya (1976), and Charnes et al. (1978). It was shown by Straffin and Heaney (1981) that the allocation method proposed by Gately corresponds precisely to the ACA-method.

The purpose of this chapter, which is based on Otten (1993), is to provide a characterization of the ACA-method on a certain class of cost games with a fixed player set as well as on a class of cost games with a variable player set, using a reduced game property. This is the subject of Section 3.3. First, in section 3.1 we formulate cost allocation problems in a game theoretic framework and in section 3.2 we recall some of the cost allocation methods proposed by the TVA.

3.1 Cost allocation problems and game theory

To formulate a cost allocation problem in terms of cooperative game theory, it is modelled as a *cost game* (N, c). Here N represents the set of participants among which the joint costs should be divided. N can be a set of potential customers of a public facility, the divisions of a firm, municipalities which share a joint water system, etc. The function $c : 2^N \to \mathbf{R}$ is the so-called *(joint) cost function.* For each coalition $S \in 2^N$, $c(S)$ denotes the minimal costs of designing a project only to serve the purposes of the members of S. Particularly, $c(\emptyset) = 0$. By CG^N we denote the class of all cost games with player set N.

Given a cost game (N, c), the cost allocation problem becomes how to allocate the joint costs in a fair way. For games corresponding to reward situations (cf. Section 2.1) notions like payoff vector, imputation set, core and so on are important. For cost games these notions should be modified in an appropriate way.

A *cost allocation for* (N, c) is a vector $x \in \mathbf{R}^N$ such that $\sum_{i \in N} x_i = c(N)$. Here, x_i is the cost allocated to player $i \in N$. (Note that we have included efficiency in the definition of a cost allocation.) If A^N is a subset of CG^N, then a *(cost) allocation method* on A^N is a map $f : A^N \to \mathbf{R}^N$, which assigns to every cost game $(N, c) \in A^N$ a cost allocation $f(c) \in \mathbf{R}^N$. The *core* of a cost game (N, c) is defined by

$$C^{cg}(c) := \{x \in \mathbf{R}^N \mid x(N) = c(N), x(S) \leq c(S) \text{ for all } S \in 2^N\}.$$

Here, the superscript *cg* indicates that we consider cost games. Further, we say that a cost game (N, c) is *concave* if and only if the (reward) game $(N, -c)$ is convex. The reader will have little difficulties in defining the Shapley value and the nucleolus for cost games. Also the τ-value can be defined for cost games. For a cost game (N, c), the τ-value is the efficient compromise between the two vectors $SC(c)$ and $m^{cg}(c)$ defined by

$$\begin{aligned} SC_i(c) &:= c(N) - c(N \setminus \{i\}) \text{ for all } i \in N \\ m_i^{cg}(c) &:= \min_{S:i\in S}(c(S) - \sum_{j\in S\setminus\{i\}} SC_j(c)) \text{ for all } i \in N. \end{aligned}$$

$SC_i(c)$ denotes the *separable cost of player i in c* (This name will be explained in Section 3.2). Note that $SC(c)$ corresponds to the vector $M(c)$ which is introduced in Section 2.1. Contrary to the reward case, however, in the context of cost allocation problems $SC(c)$ will play the role of the lower bound. Similarly, the vector $m^{cg}(c)$ is the cost game analogue of the lower value for reward games. In the context of cost allocation problems it plays the role of the upper value. Of course, the τ-value for cost games is, similar to the reward case, only defined when the vectors $SC(c)$ and $m^{cg}(c)$ constitute real lower and upper bounds, respectively. It is'easy to check that in case (N, c) is concave, we have $m_i^{cg}(c) = c(\{i\})$ for all $i \in N$, so the τ-value can easily be computed. Aoki (1989) analyses the τ-value for concave cost games.
In Tijs and Driessen (1986) the τ-value for cost games is introduced using gap functions and, instead of the name τ-value, the name *cost gap method* is used. See also Driessen (1988).

We conclude this section with an example of a special class of cost allocation problems. As in Section 2.5 we concentrate on the τ-value.

Example 3.1.1 *(Airport games (Littlechild and Owen (1973)))*
A special type of cost allocation situations is related to the aircraft landing fee problem of an airport with one runway. Suppose that the planes which are to land are classified into m types. Let N_j be the set of landings by planes of type j over a fixed period of time. Then $N := \bigcup_{j=1}^m N_j$ is the set of all landings. Let $n_j := |N_j|$ and $n := \sum_{j=1}^m n_j$.
The cost of building a runway depends on the largest plane for which the runway is designed. Let t_j be the cost to make the runway suitable for landings by planes of type j. We assume that

$$0 =: t_0 < t_1 < t_2 < \ldots < t_m.$$

To formulate this problem in a game theoretic framework we define the cost function $c : 2^N \to \mathbf{R}$ by $c(\emptyset) := 0$ and for $S \in 2^N \setminus \{\emptyset\}$

$$c(S) := \max\{t_j \mid 1 \leq j \leq m, S \cap N_j \neq \emptyset\}.$$

For the τ-value of this *airport game* we have (cf. Tijs and Driessen (1986)) in case $n_m \geq 2$

$$\tau_i(c) = t_m t_j (\sum_{k=1}^{m} n_k t_k)^{-1} \text{ if } i \in N_j.$$

So the τ-value assigns cost allocations proportional to the cost of a shortest runway needed by a player. The proof of this statement is based on the fact that airport games are concave, and that since $n_m \geq 2$, we have that the separable cost $SC_i(c)$ of each player $i \in N$ equals zero. Hence, $\tau(c)$ is proportional to $(c(\{i\}))_{i \in N}$. It can be shown that not necessarily $\tau(c) \in C^{cg}(c)$. Driessen (1988) gives an explicit formula for the τ-value in case $n_m = 1$.

In Littlechild and Owen (1973) and Dubey (1982) the Shapley value of airport games is discussed and characterized. For the nucleolus of airport games the reader is referred to Littlechild (1974), Littlechild and Owen (1976), and Owen (1982).

3.2 The TVA and cost allocation problems

TVA engineers developed several methods to allocate the costs of water projects to the purposes to be served. Almost all these methods begin by charging every player (purpose) a minimal cost, called separable cost, which are the additional cost of including the player in the project already designed for the other players. Thus, for a cost game (N, c), *the separable cost* $SC_i(c)$ *of player* $i \in N$ are defined by

$$SC_i(c) := c(N) - c(N \setminus \{i\}).$$

To use methods based on the idea above it is reasonable to make the following two assumptions on the underlying cost game:

$$SC_i(c) \leq c(\{i\}) \quad \text{for all } i \in N, \tag{3.1}$$

$$\sum_{i \in N} SC_i(c) \leq c(N) \leq \sum_{i \in N} c(\{i\}). \tag{3.2}$$

Conditions (3.1) and (3.2) are well-known balancedness conditions for cost games. If $SC_i(c) > c(\{i\})$ for some $i \in N$, then it is not favourable to include player i in the joint project. Condition (3.2) implies that after each player is charged his minimal costs there is still a nonnegative amount of cost remaining which should be allocated. These remaining cost are called the *nonseparable cost* and are given by

$$NSC(c) := c(N) - \sum_{i \in N} SC_i(c).$$

An easy way to allocate the nonseparable cost is to divide these cost equally among the players. This method is called the *egalitarian nonseparable cost* ($ENSC$) *method*, and it is one of the first allocation methods being proposed by the TVA. Thus, for a cost game (N, c) the cost allocated to player $i \in N$ by the ENSC-method are

$$ENSC_i(c) = SC_i(c) + \frac{1}{|N|} NSC(c).$$

An alternative allocation method is the *alternate cost avoided (ACA) method.* This method allocates the nonseparable cost in proportion to $c(\{i\}) - SC_i(c)$, which represents the *alternate cost avoided* by including player i in the joint project. Hence,

$$ACA_i(c) := SC_i(c) + \frac{c(\{i\}) - SC_i(c)}{\sum_{j \in N}(c(\{j\}) - SC_j(c))} NSC(c) \quad \text{for all } \ i \in N.$$

If the denominator in this formula equals 0, then $NSC(c) = 0$. So, in this case $ACA(c) := SC(c)$.

A modification of the ACA-method is the *separable cost remaining benefit (SCRB) method.* This method has become the principal method used by civil engineers to allocate the costs of multipurpose water projects (see e.g. Inter-Agency Committee on Water Resources (1958)). If $b(i)$ is the benefit of the project to player i, then i would not be willing to pay more than $\min\{b(i), c(\{i\})\}$. The *remaining benefit* to player i is defined by $\min\{b(i), c(\{i\})\} - SC_i(c)$. The SCRB-method allocates the nonseparable cost proportional to the remaining benefits. Since in many situations the benefits will exceed the alternate costs, the SCRB-method often coincides with the ACA-method.

The major drawback of the cost allocation methods mentioned above is that they only take into account the values of the coalitions with 1, $|N| - 1$, and $|N|$ players. In particular, there is no guarantee that the corresponding allocations of these methods are core elements of the cost game, which means that there might be subcoalitions that have an incentive to split off from the grand coalition.

From a practical viewpoint however, the advantage of these methods is that in general they are much easier to compute than game theoretical solution concepts as the Shapley value, the nucleolus, or the τ-value, which in principle take into account the values of *all* coalitions.

Moreover, as is shown in e.g. Suzuki and Nakayama (1976), Legros (1982), and Driessen and Tijs (1985) there are (large) classes of cost games for which some of the solution concepts mentioned above coincide with one (or more) of the game theoretical solution concepts. Particularly, for concave cost games the τ-value and the ACA-method coincide.

Example 3.2.1 An example of a cost allocation problem to be solved by TVA engineers was the apportioning of the costs of the Wilson Dam to the purposes navigation, flood control, and the provision of hydro-electric power (there were also two other, less important purposes). Here the purposes navigation, flood control and hydro-electric power are denoted as the players 1, 2, and 3, respectively. Table 3.1 is adapted from Ransmeier (1942, p. 329).

coalition S	$\emptyset$	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
cost $c(S)$	0	163,520	140,826	250,096	301,607	378,821	367,370	412,584

Table 3.1. The cost game for the Wilson Dam (costs in \$ 1,000).

For this cost game the cost allocations of the ENSC- and ACA-method are given in Table 3.2 together with the cost allocations corresponding to the game theoretical solutions mentioned above. Note that, since c is concave, the cost allocations by the ACA-method and the τ-value coincide.

	1	2	3
ENSC-method	119,424	107,973	185,187
ACA-method	117,476	99,157	195,951
Shapley value	117,829	100,756	193,999
Nucleolus	116,234	93,540	202,810
τ-value	117,476	99,157	195,951

Table 3.2. Cost allocations for the cost game of Table 3.1 (costs in \$ 1,000).

3.3 Characterizations of the ACA-method

This section further investigates the ACA-method. Attention is restricted to the class of cost games (N, c) for which (3.1) and (3.2) hold. This class is denoted by F^N and F_m denotes the class of cost games with m or more players and satisfying (3.1) and (3.2).

Geometrically, for a cost game $(N, c) \in F^N$ the cost allocation $ACA(c)$ is the unique element in the hyperplane $\{x \in \mathbf{R}^N \mid \sum_{i \in N} x_i = c(N)\}$ which lies on the line segment with endpoints $(SC_i(c))_{i \in N}$ and $(c(\{i\}))_{i \in N}$ (see Figure 3.1).

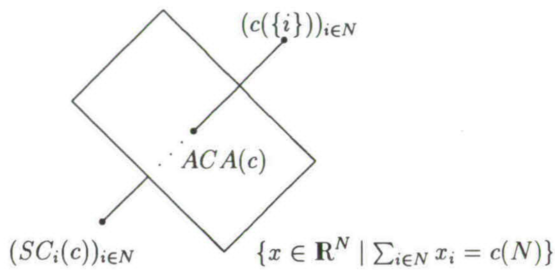

Figure 3.1.

In order to provide characterizations of the ACA-method we investigate several properties of the ACA-method which are the cost game analogies of properties introduced in Section 2.2 for values for TU-games.

Let $A \subset F_1$. Clearly, the ACA-method satisfies *individual rationality on* A, i.e., $ACA_i(c) \leq c(\{i\})$ for all $i \in N$ and all $(N, c) \in A$.

Furthermore, the ACA-method satisfies *symmetry on* A, i.e., for all $(N, c) \in A$ and all players i and j that are symmetric in (N, c), i.e., $c(S \cup \{i\}) = c(S \cup \{j\})$ for all $S \subset N \setminus \{i, j\}$, it holds that $ACA_i(c) = ACA_j(c)$.

The ACA-method also satisfies *covariance on* A, i.e., for all $(N, c) \in A$, all $k > 0$ and all $a \in \mathbf{R}^N$, such that $(N, kc + a) \in A$, we have that $ACA(kc + a) = kACA(c) + a$.

Another property of the ACA-method on A is *weak proportionality* which says that if $(N, c) \in A$ is such that $SC_i(c) = 0$ for all $i \in N$, then $ACA(c)$ is proportional to the vector $(c(\{i\}))_{i \in N}$ of individual costs.

This weak proportionality property shows great resemblance to the restricted proportionally property of the τ-value (see Theorem 2.2.2). Cost games for which each player's separable cost are zero arise when the increase in the total costs of adding an extra player can be neglected compared to the total cost of the project. An example is the airport game of Example 3.1.1 if there are at least two planes of the largest type ($n_m \geq 2$).

Similar to the characterization of the τ-value in Theorem 2.2.3 one can prove the following characterization of the ACA-method.

Theorem 3.3.1 The ACA-method is the unique cost allocation method on F^N which satisfies covariance and weak proportionality.

Proof. Suppose that $f : F^N \to \mathbf{R}^N$ satisfies the two mentioned properties. Let $(N, c) \in F^N$. It suffices to show that $f(c) = ACA(c)$.
Define the game $(N, \widehat{c}) \in F^N$ by

$$\widehat{c}(S) := c(S) - \sum_{i \in S} SC_i(c) \quad \text{for all } S \subset N.$$

Then $SC_i(\widehat{c}) = 0$ for all $i \in N$. From the weak proportionality property it follows that there exists an $\alpha \in \mathbf{R}$ such that for all $i \in N$

$$f_i(\widehat{c}) = \alpha\widehat{c}(\{i\}) = \alpha(c(\{i\}) - SC_i(c)).$$

From covariance it follows that for all $i \in N$

$$f_i(c) = SC_i(c) + f_i(\widehat{c}) = SC_i(c) + \alpha(c(\{i\}) - SC_i(c)).$$

Using the fact that $\sum_{i \in N} f_i(c) = c(N)$, it easily follows that $f(c) = ACA(c)$. □

The last part of this section provides a characterization of the ACA-method on the class F_1 using a reduced game property. In the literature several types of reduced games have been considered to provide a foundation of game theoretic solution concepts based on the consistency principle. We mention Hart and Mas-Colell (1989) for the Shapley value, Sobolev (1975), and Snijders (1991) for the (pre)nucleolus, Peleg (1986) for the core, and recently, Driessen (1993) for the τ-value. Also the ENSC-method has been characterized by means of a reduced game property (Moulin (1985), Driessen and Funaki (1993)). For detailed surveys on consistency see e.g. Thomson (1990) and Driessen (1991).
The idea behind consistency is the following. Given a cost game, and a cost allocation for this game, determined by a cost allocation method, imagine that a subcoalition decides to renegotiate the allocation within their subgroup. The new situation is described by a reduced game. A cost allocation method is called consistent with respect

to this reduced game if the method proposes in the reduced game the same outcome as in the original game for all members of this subgroup.

Let (N, c) be a cost game, $k \in N$ and $x \in \mathbf{R}^N$ a cost allocation. The *reduced game* $(N \setminus \{k\},\ c^{k,x})$ *corresponding to* (N, c) is defined as follows. For $S \subset N \setminus \{k\}$

$$c^{k,x}(S) := \begin{cases} c(S) & \text{if } |S| \leq 1 \\ c(S \cup \{k\}) - x_k & \text{if } 2 \leq |S| \leq |N| - 1. \end{cases}$$

It should be noted that the reduced game introduced here coincides with the reduced game of Moulin (1985) except for the 1-person coalitions.
The interpretation of this reduced game is as follows. In the reduced situation the cost of a 1-person coalition is the same as in the original game. However, if in the reduced situation the players want to cooperate in a coalition S, then player k should be involved in this cooperation, and therefore, the cost of coalition S in the reduced game is the cost of coalition $S \cup \{k\}$ in the original game minus the original cost x_k allocated to player k.

Let $A \subset F_1$ and let $m(A) := \min\{|N| \mid F_{|N|} \cap A \neq \emptyset\}$. A cost allocation method f on A satisfies the *reduced game property on* A if for all $(N, c) \in A$ with $|N| > m(A)$ and all $k \in N$ it holds that

(i) $(N \setminus \{k\},\ c^{k,f(c)}) \in A$, and

(ii) $f_i(c^{k,f(c)}) = f_i(c) \quad$ for all $\ i \in N \setminus \{k\}$.

The ACA-method satisfies the reduced game property on the class F_3. This is shown in the next lemma.

Lemma 3.3.2 The ACA-method satisfies the reduced game property on F_3.

Proof. Let $(N, c) \in F_3$ with $|N| \geq 4$, and let $k \in N$. We first show that the reduced game $(N \setminus \{k\}, c^{k,ACA(c)})$ is an element of F_3. For this note that for all $i \in N \setminus \{k\}$

$$c^{k,ACA(c)}(\{i\}) = c(\{i\}) \tag{3.3}$$

and since $|N| \geq 4$ also

$$SC_i(c^{k,ACA(c)}) = c(N) - ACA_k(c) - (c(N \setminus \{i\}) - ACA_k(c)) = SC_i(c). \tag{3.4}$$

Since $(N, c) \in F_3$, it follows that $SC_i(c^{k,ACA(c)}) \leq c^{k,ACA(c)}(\{i\})$ for all $i \in N \setminus \{k\}$.
It remains to show that

$$\sum_{i\in N\setminus\{k\}} SC_i(c^{k,ACA(c)}) \leq c^{k,ACA(c)}(N\setminus\{k\}) \leq \sum_{i\in N\setminus\{k\}} c^{k,ACA(c)}(\{i\}). \tag{3.5}$$

Note that for $i \in N \setminus \{k\}$

$$SC_i(c) \leq ACA(c) \leq c(\{i\}).$$

Then, using (3.3), (3.4), and the fact that $c^{k,ACA(c)}(N\setminus\{k\}) = \sum_{i\in N\setminus\{k\}} ACA_i(c)$ the required inequalities (3.5) are easily obtained.
Now we show that $ACA_i(c^{k,ACA(c)}) = ACA_i(c)$ for all $i \in N \setminus \{k\}$.
Recall that $ACA_i(c) = SC_i(c) + \alpha(c(\{i\}) - SC_i(c))$ for all $i \in N$, where α is such that

$$c(N) = \sum_{i\in N} SC_i(c) + \alpha \sum_{i\in N}(c(\{i\}) - SC_i(c)). \tag{3.6}$$

Using (3.3) and (3.4), we obtain that $ACA_i(c^{k,ACA(c)}) = SC_i(c) + \beta(c(\{i\}) - SC_i(c))$ for all $i \in N \setminus \{k\}$, where β is such that

$$c(N) - ACA_k(c) = \sum_{i\in N\setminus\{k\}} SC_i(c) + \beta \sum_{i\in N\setminus\{k\}} (c(\{i\}) - SC_i(c)). \tag{3.7}$$

Subtracting (3.7) from (3.6) we obtain

$$ACA_k(c) = SC_k(c) + \alpha(c(\{k\}) - SC_k(c)) + (\alpha - \beta) \sum_{i\in N\setminus\{k\}} (c(\{i\}) - SC_i(c)).$$

Hence,

$$(\alpha - \beta) \sum_{i\in N\setminus\{k\}} (c(\{i\}) - SC_i(c)) = 0. \tag{3.8}$$

We now distinguish two cases.
If $\sum_{i\in N\setminus\{k\}}(c(\{i\}) - SC_i(c)) = 0$, then $c(\{i\}) - SC_i(c) = 0$ for all $i \in N \setminus \{k\}$ and hence, $ACA_i(c) = SC_i(c) = ACA_i(c^{k,ACA(c)})$ for all $i \in N \setminus \{k\}$.
If $\sum_{i\in N\setminus\{k\}}(c(\{i\}) - SC_i(c)) \neq 0$, then by (3.8) $\alpha - \beta = 0$. Hence, $ACA_i(c^{k,ACA(c)}) = ACA_i(c)$ for all $i \in N \setminus \{k\}$. □

Example 3.3.3 illustrates that the ACA-method does not satisfy the reduced game property on the set F_2. This is due to the fact that by reducing a 3-person game to a 2-person game the separable costs of the players may change.

Example 3.3.3 Let $N := \{1,2,3\}$ and define (N,c) as follows. For $S \subset N$

$$c(S) = \begin{cases} 2 & \text{if } \{2,3\} \not\subset S \\ 4 & \text{if } \{2,3\} \subset S. \end{cases}$$

Clearly, $(N,c) \in F_2$ and $ACA(c) = (0,2,2)$. The reduced game $(\{1,2\}, c^{3,ACA(c)}) \in F_2$ is given by $c^{3,ACA(c)}(\{1\}) = c^{3,ACA(c)}(\{2\}) = 2$ and $c^{3,ACA(c)}(\{1,2\}) = 2$.
Hence, $ACA(c^{3,ACA(c)}) = (1,1) \neq (0,2) = (ACA_1(c), ACA_2(c))$.

The following lemma provides a relation between the reduced game property and weak proportionality.

Lemma 3.3.4 Let f be a cost allocation method on F_3 which satisfies weak proportionality on $F_3 \setminus F_4$ and the reduced game property on F_3. Then f satisfies weak proportionality on F_3.

Proof. The proof proceeds by induction on the number of players.
Let $(N,c) \in F_3$ with $|N| \geq 4$ be such that $SC_i(c) = 0$ for all $i \in N$ and suppose that f satisfies weak proportionality on $F_3 \setminus F_{|N|}$.
Let $k \in N$ and let $(N \setminus \{k\}, c^{k,f(c)})$ be the $(|N|-1)$-person reduced game of (N,c). Then $(N \setminus \{k\}, c^{k,f(c)}) \in F_3 \setminus F_{|N|}$. Since $SC_i(c^{k,f(c)}) = 0$ for all $i \in N \setminus \{k\}$ (cf. (3.4)), it follows from the induction hypothesis that there exists an $\alpha (= \alpha_k) \in \mathbf{R}$ such that

$$f_i(c^{k,f(c)}) = \alpha c^{k,f(c)}(\{i\}) = \alpha c(\{i\}) \quad \text{for all } \; i \in N \setminus \{k\}.$$

Since f satisfies the reduced game property on F_3 it follows that

$$f_i(c) = f_i(c^{k,f(c)}) = \alpha c(\{i\}) \quad \text{for all } \; i \in N \setminus \{k\}.$$

If we vary $k \in N$, it follows that α does not depend on k. Hence, we obtain

$$f(c) = \alpha(c(\{1\}), \ldots, c(\{n\})),$$

and so f satisfies weak proportionality on $F_3 \setminus F_{|N|+1}$. □

Now we can formulate the main theorem of this chapter which characterizes the ACA-method on F_1.

Theorem 3.3.5 The ACA-method is the unique cost allocation method on F_1 which satisfies

(i) symmetry on F_1,

(ii) covariance on F_1,

(iii) weak proportionality on $F_3 \setminus F_4$,

(iv) the reduced game property on F_3.

Proof. Clearly, the ACA-method satisfies (i)-(iv).
Let f be a cost allocation method, defined on F_1, satisfying (i)-(iv). Let $(N,c) \in F_1$. To show that $f(c) = ACA(c)$ we distinguish three cases.
If $|N| = 1$, then $f(c) = c(\{1\}) = ACA(c)$.
If $|N| = 2$, then (i) and (ii) imply that $f_i(c) = c(\{i\}) + \frac{1}{2}(c(N) - c(\{i\}) - c(N \setminus \{i\})) = ACA_i(c)$ for $i = 1, 2$.
If $|N| \geq 3$, then Theorem 3.3.1 and Lemma 3.3.4 imply $f(c) = ACA(c)$. □

It may be noted that also the ENSC-method satisfies symmetry, covariance, and the reduced game property on the set F_3. However, this cost allocation method does not satisfy weak proportionality.

For a cost game $(N,c) \in F_1$, the *center of imputation set (CIS) value* (cf. Driessen and Funaki (1993)) is defined by

$$CIS_i(c) := c(\{i\}) + \frac{1}{|N|}(c(N) - \sum_{j \in N} c(\{j\})) \quad \text{for all } i \in N.$$

If in Theorem 3.3.5 condition (iii) is omitted and condition (iv) is replaced by the reduced game property on F_2 then a characterization of the CIS-value on F_1 is obtained. It is left to the reader to show that the CIS-value is indeed the unique cost allocation method on F_1 which satisfies symmetry and covariance F_1 and the reduced game property on F_2.

Chapter 4

The compromise value for NTU-games

In Section 2.4 we defined the compromise value for NTU-games and showed that the compromise value by definition extends the τ-value for TU-games and the Raiffa-Kalai-Smorodinsky solution (RKS-solution) for bargaining problems to NTU-games (Corollary 2.4.5). We have also seen that the characterizations of the τ-value and the RKS-solution can be generalized to NTU-games to obtain characterizations of the compromise value (Theorem 2.4.6 and Theorem 2.4.7 (or 2.4.9)).
In this chapter, based on Otten, Borm and Tijs (1994), we reconsider the compromise value. In (most of) the characterizations of the compromise value discussed in Section 2.4 a non-levelness condition plays a crucial role. Section 4.1 illustrates that this condition can be weakened in order to obtain a characterization on a larger class of NTU-games. We use a similar technique as Peters and Tijs (1984) who extended Thomson's (1980) characterization of the RKS-solution to a larger class of bargaining problems by weakening the non-levelness condition.
Further, Section 4.2 characterizes the set of all monotonic, Pareto optimal, and covariant values on this class of NTU-games using monotonic curve solutions as introduced by Peters and Tijs (1984).

4.1 Characterizations of the compromise value

In Section 2.4 we have seen that the compromise value satisfies weak Pareto optimality on the class C^N of compromise admissible NTU-games. In this chapter we focus attention on subclasses of C^N for which this property can be strengthened. First, we introduce some notation.

Let (N, V) be an NTU-game. For each $S \in 2^N \setminus \{\emptyset\}$, let $wdom(V(S))$ denote the set of elements which are *weakly dominated by coalition* S, i.e.,

$$wdom(V(S)) := \{x \in \mathbf{R}^S \mid x \leq y \text{ for some } y \in V(S),\ y \neq x\}.$$

Clearly, $dom(V(S)) \subset wdom(V(S))$.

For what follows the following additional property for values for NTU-games is important.

Let $A^N \subset C^N$, and let $f : A^N \to \mathbf{R}^N$ be a value on A^N.

f is called *Pareto optimal* on A^N if $f(V) \in V(N) \setminus wdom(V(N))$ for all $V \in A^N$.

Clearly, Pareto optimality is a strengthening of the weak Pareto optimality condition introduced in Section 2.4. On the class C^N of compromise admissible NTU-games the compromise value does not satisfy Pareto optimality. This is shown in the following example.

Example 4.1.1 Let $N := \{1, 2, 3\}$ and define V by
$V(S) := \{x \in \mathbf{R}^S \mid x \leq 0\}$ for all $S \in 2^N \setminus \{\emptyset, N\}$, and
$V(N) := comp(conv\{(4,0,0),\ (4,3,0),\ (2,4,0),\ (0,4,0),\ (2,3,2),\ (0,3,2),\ (0,0,4)\})$.
The reader easily verifies that $K(V) = (4,4,4)$ and $k(V) = (0,0,0)$. So, $V \in C^N$ and $T(V) = (2,2,2)$. But $(2,2,2) \in wdom(V(N))$ since $(2,3,2) \in V(N)$. Hence, the compromise value is not Pareto optimal on C^N.

Of course, in the characterizations of the compromise value in Section 2.4 weak Pareto optimality can be replaced by Pareto optimality since, by (2.1), for a game $V \in \overline{C}^N$ (or $\widetilde{C}^N$) all weakly Pareto optimal points in the set $V^*(N)$ are Pareto optimal.

In this section we will show that by modifying the non-levelness assumption one can obtain a characterization of the compromise value on a larger class of compromise admissible NTU-games. This modification is based on Peters and Tijs (1984), who extended Thomson's (1980) characterization of the RKS-solution to a larger class of bargaining problems by weakening the assumption of non-levelness.

We restrict attention to the class $\widehat{C}^N$ of all compromise admissible NTU-games V with player set N satisfying (2.2)-(2.4) and, in addition,

for all $x \in V^*(N)$ and all $i \in N$ we have: If $x \in wdom(V(N))$ and $x_i < K_i(V)$, then there exists an $\epsilon > 0$ such that $x + \epsilon e^i \in V(N)$. (4.1)

Clearly, if $V^*(N)$ is non-level, then $V^*(N)$ also satisfies (4.1).
Note that the NTU-game provided in Example 4.1.1 does not satisfy (4.1). This is an immediate consequence of the following lemma which shows that the compromise value is Pareto optimal on the class $\widehat{C}^N$.

Lemma 4.1.2 Let $V \in \widehat{C}^N$. Then $T(V) \in V(N) \setminus wdom(V(N))$.

Proof. Because of covariance of T attention can be restricted to $V \in \widehat{C}^N$ with $k(V) = 0$ and $K(V) = e^N$ (see the proof of Theorem 2.4.8). So, let $V \in \widehat{C}^N$ with $k(V) = 0$ and $K(V) = e^N$. Then $T(V)$ is an element of the line segment through 0 and e^N. To prove that $T(V) \in V(N) \setminus wdom(V(N))$ we distinguish two cases.
Obviously, if $T(V) = e^N$, then $T(V) \in V(N) \setminus wdom(V(N))$. Now suppose that $T(V) \neq e^N$ and that $T(V) \in wdom(V(N))$. Then $T(V) < e^N = K(V)$, and so by (4.1), it follows that for each $i \in N$ there exists an $\epsilon_i > 0$ such that $T(V) + \epsilon_i e^i \in V(N)$. Take $\epsilon := \min\{\epsilon_i \mid i \in N\}$. By comprehensiveness of $V(N)$ it follows that $T(V) + \epsilon e^i \in V(N)$ for all $i \in N$. Using convexity of $V(N)$ we obtain that $T(V) + \frac{\epsilon}{|N|} e^N \in V(N)$. Hence, $T(V) \in dom(V(N))$, which contradicts the weak Pareto optimality of T. Hence, $T(V) \in V(N) \setminus wdom(V(N))$. □

The following theorem extends Theorem 2.4.9 to the class $\widehat{C}^N$.

Theorem 4.1.3 The compromise value is the unique value on $\widehat{C}^N$ which satisfies

(i) Pareto optimality,

(ii) symmetry,

(iii) monotonicity,

(iv) covariance.

Proof. (The proof of this theorem follows the same line as the proof of Theorem 2.4.7 by Borm et al. (1992).) Clearly, the compromise value satisfies the four properties mentioned above on $\widehat{C}^N$. Now let $f : \widehat{C}^N \to \mathbf{R}^N$ satisfy the four properties. We prove that $f(V) = T(V)$ for all $V \in \widehat{C}^N$.
Because of covariance of f and T it is sufficient to prove that $f(V) = T(V)$ for all $V \in \widehat{C}^N$ with $k(V) = 0$ and $K(V) = e^N$ (see the proof of Theorem 2.4.8). So, let $V \in \widehat{C}^N$ with $k(V) = 0$ and $K(V) = e^N$. Then $T(V)$ is an element of the line segment through 0 and e^N. Using the assumptions (2.3) and (2.4) we have that $conv\{e^i \mid i \in N\} \subset V(N)$, so $T(V) \geq \frac{1}{|N|} e^N$.
Now consider the NTU-game (N, W) defined by

$$W(S) := \begin{cases} \{x \in \mathbf{R}^S \mid x \leq 0\} & \text{if } S \in 2^N \setminus \{\emptyset, N\} \\ comp(conv(\{e^i \mid i \in N\} \cup \{T(V)\})) & \text{if } S = N. \end{cases}$$

Obviously, $K(W) = e^N$, and $k(W) = 0$. Hence, $W \in C^N$ and assumptions (2.2)-(2.4) are satisfied. If $T(V) = e^N$, then $W(N) = comp\{e^N\}$. Otherwise, if $T(V) < e^N$, then $W(N)$ is non-level. In both cases (4.1) is satisfied, so $W \in \widehat{C}^N$. Clearly, $T(W) = T(V)$. Using symmetry of f it follows that $f_i(W) = f_j(W)$ for all $i, j \in N$. So, by Pareto optimality of f and T it follows that $f(W) = T(W)$. Hence, $T(V) = f(W)$. Since, $W(N) \subset V(N)$, $k(V) = k(W)$, and $K(V) = K(W)$, it follows by monotonicity of f that $f(W) \leq f(V)$. Hence, $T(V) \leq f(V)$. But then Pareto optimality of T implies that $T(V) = f(V)$. □

4.2 Monotonic, Pareto optimal, and covariant values

Theorem 4.1.3 characterizes the compromise value as the unique value on $\widehat{C}^N$ which satisfies Pareto optimality, monotonicity, covariance and symmetry. In this section we drop the symmetry property and characterize all Pareto optimal, monotonic and covariant solutions on the class $\widehat{C}^N$. For this, we use similar techniques as Peters and Tijs (1984) who characterized all Pareto optimal, monotonic, and covariant bargaining solutions on a large class of bargaining problems, using monotonic curve solutions.

As we consider covariant values on $\widehat{C}^N$ attention can be restricted to the class $\widehat{C}^N_{0,1}$ of NTU-games $V \in \widehat{C}^N$ which satisfy $K(V) = e^N$ and $k(V) = 0$ (cf. the proof of Theorem 4.1.3).

Using monotonic curves one can define monotonic and Pareto optimal values on the class $\widehat{C}^N_{0,1}$.

A *monotonic curve* (Peters and Tijs (1984)) is a map $\gamma : [1, |N|] \to [0, 1]^N$ with

(i) γ is increasing, i.e., $\gamma(s) \geq \gamma(t)$ if $s \geq t$, and

(ii) $\sum_{i \in N} \gamma_i(t) = t$ for all $t \in [1, |N|]$.

Note that (ii) implies that $\gamma(1) \in conv\{e^i \mid i \in N\}$, and $\gamma(|N|) = e^N$. Moreover, it can be checked that each monotonic curve is continuous.

Let γ be a monotonic curve. Then γ gives rise to a value f^γ on $\widehat{C}^N_{0,1}$ in the following way: For $V \in \widehat{C}^N_{0,1}$ define $f^\gamma(V)$ as the unique Pareto optimal point of $V(N)$ lying on the curve $\{\gamma(t) \mid 1 \leq t \leq |N|\}$. It can easily be verified that f^γ is well-defined on $\widehat{C}^N_{0,1}$ (cf. Peters and Tijs (1984)). f^γ is called *the value corresponding to the monotonic curve* γ. Clearly, f^γ is monotonic and Pareto optimal on $\widehat{C}^N_{0,1}$ and f^γ can be extended to a monotonic, Pareto optimal and covariant value on $\widehat{C}^N$ in a unique way.

We now have the following characterization.

Theorem 4.2.1 Let $f : \widehat{C}^N \to \mathbf{R}^N$ be a value on $\widehat{C}^N$. Then f satisfies Pareto optimality, monotonicity and covariance if and only if $f = f^\gamma$ for some monotonic curve $\gamma : [1, |N|] \to [0,1]^N$.

Proof. Clearly, if $f = f^\gamma$ for some monotonic curve γ, then f satisfies the required properties. Conversely, let f satisfy Pareto optimality, monotonicity and covariance. We construct $\gamma : [1, |N|] \to [0,1]^N$ as follows.
For $t \in [1, |N|]$, let $\gamma(t) := f(V_t)$, where V_t is the NTU-game defined by

$$V_t(S) := \begin{cases} \{x \in \mathbf{R}^S \mid x \leq 0\} & \text{if } S \in 2^N \setminus \{\emptyset, N\} \\ comp(\{x \in \mathbf{R}^N \mid 0 \leq x \leq e^N,\ \sum_{i \in N} x_i \leq t\}) & \text{if } S = N. \end{cases}$$

The reader easily verifies that $K(V_t) = e^N$, $k(V_t) = 0$, and $V_t \in \widehat{C}^N$ for every $t \in [1, |N|]$. Further, by Pareto optimality and monotonicity of f it follows that γ satisfies (i) and (ii). So γ is a monotonic curve. By definition, one has

$$f(V_t) = f^\gamma(V_t) \quad \text{for all } t \in [1, |N|]. \tag{4.2}$$

We want to prove that $f = f^\gamma$. In view of covariance of f and f^γ it is sufficient to prove that $f(V) = f^\gamma(V)$ for all $V \in \widehat{C}^N$ with $K(V) = e^N$ and $k(V) = 0$.
Let $V \in \widehat{C}^N$ satisfy $K(V) = e^N$ and $k(V) = 0$. Let $t := \sum_{i \in N} f^\gamma_i(V)$, and let W be the NTU-game defined by

$$W(S) := \begin{cases} \{x \in \mathbf{R}^S \mid x \leq 0\} & \text{if } S \in 2^N \setminus \{\emptyset, N\} \\ V(N) \cap V_t(N) & \text{if } S = N. \end{cases}$$

Then $W \in \widehat{C}^N$ and $K(W) = e^N$ and $k(W) = 0$. Clearly, $f^\gamma(W) = f^\gamma(V) = f^\gamma(V_t)$. Hence, by (4.2)

$$f^\gamma(V) = f(V_t). \tag{4.3}$$

Using monotonicity of f, we have $f(W) \leq f(V_t)$, and $f(W) \leq f(V)$, and by Pareto optimality of f it follows that

$$f(W) = f(V_t) = f(V). \tag{4.4}$$

Combining (4.3) and (4.4) we can conclude that $f(V) = f^{\gamma}(V)$. □

From the proof of Theorem 4.2.1 it follows that there exists a unique monotonic curve $\gamma^* : [1, |N|] \to [0,1]^N$ such that f^{γ^*} is symmetric, namely,

$$\gamma^*(t) := \frac{t}{|N|} e^N \quad \text{for all } t \in [1, |N|].$$

Clearly, $f^{\gamma^*} = T$, so Theorem 4.2.1 provides an alternative proof of Theorem 4.1.3.

Chapter 5

The MC-value for monotonic NTU-games

In Chapters 2 and 4 we studied the compromise value for NTU-games which is based on ideas underlying the τ-value for TU-games. We showed that the compromise value can be considered as an extension of the τ-value for TU-games and the RKS-solution for bargaining problems.

The idea of generalizing a solution concept for TU-games to the class of NTU-games is not new. In fact, most solution concepts for NTU-games are based on well-known solutions for TU-games. For example, the Shapley NTU-value is an extension of the Shapley value for TU-games and the Nash bargaining solution to NTU-games. Based on this fact Aumann (1985a) developed a characterization of the Shapley NTU-value having in mind the characterizations of the Shapley value (Shapley (1953)) and the Nash solution (Nash (1950)). For the Harsanyi value (Harsanyi (1963)) a similar reasoning can be followed.

An alternative way to extend the Shapley value to NTU-games was introduced in Maschler and Owen (1989, 1992). Their consistent Shapley value is a solution concept based on the following idea: First, the notion of the Shapley value is extended in a straightforward way to so-called hyperplane games, and, based on this extension, a value for general NTU-games is defined by associating hyperplane games to a general NTU-game.

In this chapter, which is based on Otten, Borm, Peleg and Tijs (1994), we introduce a new single-valued solution concept for monotonic 0-normalized NTU-games, the marginal based compromise value, or shortly, the MC-value. The idea behind the

MC-value lies in the fact that the Shapley value for TU-games can be considered as the efficient compromise between 0 and the sum of the marginal vectors (cf. Section 2.1). We extend the notion of marginal vectors to monotonic NTU-games and then define the MC-value as a straightforward generalization of the Shapley value. Consequently, the MC-value by definition extends the Shapley value, but it turns out that the MC-value also extends the RKS-solution for bargaining problems. Further, we provide two characterizations of the MC-value. The second one illustrates that the characterization of the RKS-solution by Kalai and Smorodinsky (1975) can be adjusted to provide a characterization of the MC-value. It remains an open problem, however, whether a characterization of the MC-value can be developed based on characterizations of the Shapley value (cf. Shapley (1953), Young (1985)).
The characterizations of the MC-value reflect the similarity between the MC-value and the compromise value, but another candidate for a comparison with the MC-value is the egalitarian solution introduced by Kalai and Samet (1985). Both solution concepts are defined and characterized for NTU-games which are not necessarily convex valued, and both solution concepts extend the Shapley value to NTU-games. Further, for both solution concepts the outcome is determined by 0 as a starting point and a vector which indicates the direction to move in order to obtain a weak Pareto optimal outcome. A major difference, however, is that for the egalitarian solution this direction is fixed, whereas for the MC-value the direction depends on the game, which seems to be much more natural.

This chapter is organized as follows.
In Section 5.1 we start with some basic definitions. Marginal vectors for monotonic NTU-games are defined as an extension of the marginals for TU- and hyperplane games.
In Section 5.2 we examine relations between the core of NTU-games and the marginal vectors in the spirit of the well-known relations for TU-games. It is shown that also in the NTU-case there are relations between the marginals and the core, but not all results for the TU-case can be extended to the NTU-case.
In Section 5.3 the MC-value is introduced, and it is shown that the MC-value extends the Shapley value for TU-games, the consistent Shapley value for hyperplane games, and the RKS-solution for bargaining problems to the general class of NTU-games.
Section 5.4 discusses several properties of the MC-value and yields two characterizations of the MC-value on large subclasses of NTU-games. Also a comparison between

the MC-value, the compromise value, and the egalitarian solution is provided.

5.1 NTU-games and marginal vectors

In Section 2.4 we defined an NTU-game to be a pair (N, V). Throughout this chapter we will assume that the players are indexed by the natural numbers $1, \dots, n$, where $n = |N|$. So $N = \{1, \dots, n\}$. Moreover, we will assume that V satisfies

(i) *0-normalization*: $V(\{i\}) = \{x \in \mathbf{R}^i \mid x \leq 0\}$ for all $i \in N$,

(ii) *monotonicity*: For all $S, T \in 2^N$, with $\emptyset \neq S \subset T$, and all $x \in V(S)$, there exists a $y \in V(T)$ such that $y_S \geq x$ (i.e., the projection of $V(T)$ on $\mathbf{R}^S$ contains the set $V(S)$).

These conditions are not very restrictive. (i) is imposed only for the sake of convenience. Condition (ii) implies that larger coalitions can obtain at least as much as smaller coalitions. Note that (i) and (ii) imply that the set $V_0(S) := \{x \in V(S) \mid x \geq 0\}$ is non-empty for each $S \in 2^N \setminus \{\emptyset\}$. The class of monotonic 0-normalized NTU-games with player set $N = \{1, \dots, n\}$ is denoted by Γ_m^N.

Let $V \in \Gamma_m^N$ be an NTU-game and let $\sigma \in \Pi(N)$. The *marginal vector* $m^\sigma(V)$ is defined by

$$m^\sigma_{\sigma(i)}(V) := \max\{t \in \mathbf{R} \mid (m^\sigma_{\sigma(1)}, \dots, m^\sigma_{\sigma(i-1)}, t) \in V(\{\sigma(1), \dots, \sigma(i)\})\}$$

for all $i \in N$. If there is no confusion about the game V we write m^σ instead of $m^\sigma(V)$.

Note that the marginal vectors are well-defined, because of the definition of NTU-games and monotonicity. It is also clear that $m^\sigma \in V(N) \setminus dom(V(N))$ and $m^\sigma \geq 0$ for all $\sigma \in \Pi(N)$.

This definition of marginal vectors is a straightforward extension of the notion of marginal vectors for TU-games (cf. Section 2.1). Also in the field of NTU-games marginal vectors are known: Maschler and Owen (1989) introduced marginal vectors for hyperplane games (NTU-games where for each coalition S the boundary of the set $V(S)$ is a hyperplane) to define the consistent Shapley value on this subclass of NTU-games. Our definition also extends Maschler and Owen's definition.

For the sake of completeness we give the interpretation of the marginal vector m^σ.

If $\sigma(1), \ldots, \sigma(n)$ is a certain order on the players, then m^σ assigns to player $\sigma(1)$ the maximum he can obtain in $V(\{\sigma(1)\})$. $m^\sigma_{\sigma(2)}$ is the maximum player $\sigma(2)$ can get in $V(\{\sigma(1), \sigma(2)\})$ given that he should guarantee player $\sigma(1)$ a payoff of $m^\sigma_{\sigma(1)}$, etc. So the marginal vector m^σ assigns to each player the maximum he can get if he should guarantee his predecessors the payoffs already given to them.

5.2 Marginal vectors and the core

It is well-known that for TU-games the core is contained in the convex hull of the marginal vectors (Weber (1988), cf. Derks (1992)). Hence, for TU-games this so-called 'Weber set' is a 'core catcher'. Shapley (1971) proved that for convex TU-games the core coincides with the convex hull of the marginal vectors. In this section we will examine relations between the marginal vectors and the core for monotonic NTU-games.

In the context of NTU-games there are at least two notions of convexity, ordinal convexity (Vilkov (1977)) and cardinal convexity (Sharkey (1982)). Both ordinal and cardinal convexity are extensions of the notion of convexity for TU-games. Although for convex TU-games all marginal vectors are core elements, this result can not be extended to the NTU case: There are counterexamples both for ordinal and cardinal convex NTU-games.[1]

To examine possible relations between marginal vectors and the core for the general class of monotonic NTU-games, we will consider a simple class of NTU-games, the so-called 1-corner games.

An NTU-game $V \in \Gamma^N$ is called a *1-corner game* if for all $S \in 2^N \setminus \{\emptyset\}$ there exists an element $u^S \in \mathbf{R}^S$ such that

$$V(S) = comp(u^S) = \{x \in \mathbf{R}^S \mid x \leq u^S\}.$$

Clearly, a 1-corner game V is monotonic if and only if for all $\emptyset \neq S \subset T$ and all $i \in S$, $u^S_i \leq u^T_i$. The reader easily verifies that $u^N \in C(V)$ for monotonic 1-corner games. The following proposition shows that for monotonic, 0-normalized 1-corner games all marginal vectors are core elements.

[1] A counterexample for ordinal convex NTU-games has been provided by Sjaak Hurkens (private communication).

Proposition 5.2.1 Let $V \in \Gamma_m^N$ be a 1-corner game. Then

$$C(V) = \bigcup_{\sigma \in \Pi(N)} \{x \in V(N) \mid x \geq m^\sigma\}.$$

Proof. Let $x \in C(V)$. We construct a $\sigma \in \Pi(N)$ with $x \geq m^\sigma$. Since $x \in C(V)$, there exists an $i_n \in N$ such that $x_{i_n} = u_{i_n}^N$. Define $\sigma(n) := i_n$. Since $x_{N \setminus \{i_n\}} \notin dom(V(N \setminus \{i_n\}))$, there exists an $i_{n-1} \in N \setminus \{i_n\}$ such that $x_{i_{n-1}} \geq u_{i_{n-1}}^{N \setminus \{i_n\}}$. Define $\sigma(n-1) := i_{n-1}$. By proceeding in this way we obtain a permutation σ defined by $\sigma(1) = i_1, \sigma(2) = i_2, \ldots, \sigma(n) = i_n$ such that

$$m_{i_1}^\sigma = u_{i_1}^{\{i_1\}} \leq x_{i_1}, \ldots, m_{i_{n-1}}^\sigma = u_{i_{n-1}}^{N \setminus \{i_n\}} \leq x_{i_{n-1}}, m_{i_n}^\sigma = u_{i_n}^N = x_{i_n}.$$

Hence, $x \geq m^\sigma$.
To prove the converse inclusion it is sufficient to show that $m^\sigma \in C(V)$ for all $\sigma \in \Pi(N)$. Let $\sigma \in \Pi(N)$. Without loss of generality we assume that σ is the identical permutation, i.e., $\sigma(i) = i$ for all $i \in N$. Trivially, $m^\sigma \leq u^N$, and $m_n^\sigma = u_n^N$ by definition of m^σ. Monotonicity of V implies that $m_n^\sigma \geq u_n^S$ for all $S \subset N$ with $n \in S$. Hence, $m_S^\sigma \notin dom(V(S))$ for all $S \subset N$ with $n \in S$. So the only coalitions for which $m_S^\sigma \in dom(V(S))$ is possible are coalitions $S \subset N \setminus \{n\}$. Since, $m_{n-1}^\sigma = u_{n-1}^{N \setminus \{n\}} \geq u_{n-1}^T$ for all $T \subset N \setminus \{n\}$ with $n-1 \in T$, it follows that there are no coalitions $T \subset N \setminus \{n\}$ with $n-1 \in T$ such that $m_T^\sigma \in dom(V(T))$. Repeating this argument leads to the conclusion that $m_S^\sigma \notin dom(V(S))$ for all $S \subset N$, $S \neq \emptyset$. Hence, $m^\sigma \in C(V)$. □

Proposition 5.2.1 gives rise to the following definition of a possible 'core catcher' for monotonic NTU-games, which can be considered as a generalization of the 'Weber set' to the class of NTU-games.
Let $V \in \Gamma_m^N$. The *Weber set* of V is the set

$$W(V) = \{x \in V(N) \setminus dom(V(N)) \mid \text{ there exists a } y \in conv(\{m^\sigma \mid \sigma \in \Pi(N)\}) \text{ with } x \geq y\}.$$

Clearly, this definition extends the Weber set for TU-games to the NTU-case. As Proposition 5.2.1 shows $W(V)$ is a core catcher for 1-corner games. However, for monotonic NTU-games where $V(N)$ is not convex the Weber set need not be a core catcher. Consider for example, the 2-person NTU-game defined by
$V(\{i\}) := \{x \in \mathbf{R}^i \mid x \leq 0\}, \qquad i = 1,2,$
$V(\{1,2\}) := comp(\{(1,0),(0,1)\})$.

Then $(0,0) \in C(V)$, but $(0,0) \notin W(V) = \{(1,0),(0,1)\}$.
It is an open problem whether $C(V) \subset W(V)$ for all NTU-games $V \in \Gamma_m^N$ for which $V(N)$ is a convex set. Obviously, this holds for bargaining problems.

5.3 The MC-value

In this section we will introduce a new single-valued solution concept for monotonic NTU-games based on the marginal vectors, the MC-value. Before we introduce the MC-value, we first recall that for TU-games the Shapley value can be viewed as a compromise value: For a game $v \in G^N$ it is the efficient combination of $0 \in \mathbf{R}^N$ and the sum of the marginal vectors (if $v(N) \neq 0$). This observation leads us to the following definition.

Let $V \in \Gamma_m^N$ be a monotonic and 0-normalized NTU-game. Denote

$$b(V) := \sum_{\sigma \in \Pi(N)} m^\sigma(V).$$

The *marginal based compromise value* of V, or shortly, the *MC-value* of V is the largest combination of 0 and $b(V)$ which is an element of $V(N) \setminus dom(V(N))$. Formally,

$$MC(V) := \alpha_V b(V),$$

where

$$\alpha_V := \max\{\alpha \in \mathbf{R}_+ \mid \alpha b(V) \in V(N)\}.$$

The MC-value is well-defined since $V_0(N)$ is non-empty and compact and the vector $b(V)$ is nonnegative (note that $b(V) = 0$ if and only if $V_0(S) = \{0\}$ for all $S \subset N$, $S \neq \emptyset$). The vector $b(V)$ can be regarded as an upper value for V.[2]
Clearly, the MC-value is a generalization of the Shapley value to the class of monotonic, 0-normalized NTU-games. The following theorem shows that the MC-value not only extends the Shapley value but also the consistent Shapley value for hyperplane games and the RKS-solution for bargaining problems.

[2]The 0-normalization we imposed is not very restrictive: The MC-value can be extended to a covariant value on the class of all monotonic games in the following way: Translate an arbitrary monotonic game into a 0-normalized game, compute the MC-value for this game, and then translate it back to obtain a solution for the original game.

Theorem 5.3.1

(i) On the class of monotonic, 0-normalized TU-games the MC-value coincides with the Shapley value.

(ii) On the class of monotonic, 0-normalized hyperplane games the MC-value coincides with the consistent Shapley value.

(iii) On the class of bargaining problems with disagreement outcome 0 the MC-value coincides with the RKS-solution.

Proof. The proofs of (i) and (ii) are straightforward consequences of the definition and therefore we only prove (iii).
Let $V \in \Gamma_m^N$ be the NTU-game corresponding to the bargaining problem $(C,0)$ and let $\sigma \in \Pi(N)$ be a permutation. Then the marginal vector m^σ satisfies

$$m^\sigma_{\sigma(i)} = \begin{cases} u_{\sigma(n)}(C,0), & \text{if } i = n, \\ 0, & \text{if } i \neq n. \end{cases}$$

As a consequence it follows that $b(V) = (n-1)!u(V)$, and hence $MC(V) = RKS(V)$. □

We conclude this section with an example.

Example 5.3.2 Consider the NTU-game $(\{1,2,3\}, V_\varepsilon)$ corresponding to the exchange market discussed in Example 2.5.6. Note that V_ε is not 0-normalized if $\epsilon \neq 0$. If we compute the MC-value of this game by following the approach described in footnote 2 on the previous page, we obtain

$$MC(V_\varepsilon) = (\frac{5}{12} - \frac{5}{12}\epsilon, \frac{5}{12} - \frac{5}{12}\epsilon, \frac{1}{6} + \frac{5}{6}\epsilon).$$

In Example 2.5.6 we have seen that this outcome is also prescribed by the Shapley NTU-value.

5.4 Characterizations of the MC-value

In this section we investigate several properties of the MC-value and moreover, two characterizations of the MC-value are provided. We conclude this section with a comparison between the MC-value and the egalitarian solution.

Some properties of the MC-value are summarized in Proposition 5.4.1.

Proposition 5.4.1 On the class Γ_m^N of monotonic NTU-games the MC-value satisfies the following properties.

(i) *weak Pareto optimality:* $MC(V) \in V(N) \setminus dom(V(N))$ for all $V \in \Gamma_m^N$.

(ii) *scale covariance:* $MC(\lambda * V) = \lambda * MC(V)$ for all $\lambda \in \mathbf{R}_{++}^N$ and all $V \in \Gamma_m^N$.

(iii) *symmetry:* $MC_i(V) = MC_j(V)$ for all $V \in \Gamma_m^N$ and all $i, j \in N$ which are symmetric in V (cf. Section 2.4).

(iv) *b-symmetry:* $MC_i(V) = MC_j(V)$ for all $i, j \in N$ with $b_i(V) = b_j(V)$, and all $V \in \Gamma_m^N$.

(v) *conditional monotonicity:* $MC(V) \leq MC(W)$ for all NTU-games $V, W \in \Gamma_m^N$ with $V(N) \subset W(N)$, $b(W) > 0$, and $\frac{b_i(V)}{b_i(W)} = \frac{b_j(V)}{b_j(W)}$ for all $i, j \in N$.

Proof. We will only prove (ii). The other properties are obvious. Let $V \in \Gamma_m^N$, $\lambda \in \mathbf{R}_{++}^N$ and $\sigma \in \Pi(N)$. The reader easily verifies that $m^\sigma(\lambda * V) = \lambda * m^\sigma(V)$, and so $b(\lambda * V) = \lambda * b(V)$. From this it immediately follows that $MC(\lambda * V) = \lambda * MC(V)$. □

Besides the properties(i)-(iii) the MC-value satisfies also other standard properties as individual rationality and the null player (or dummy player) property. Property (iv) is a stronger version of symmetry. It is introduced to characterize the MC-value and states that if for two players in a game the sum of all their marginal contributions is equal, then they should get the same payoff. Property (v) strengthens the (restricted) monotonicity which is used to characterize the RKS-solution for bargaining problems. The interpretation is that if the set of attainable payoffs for the grand coalition becomes larger and the direction of the upper value does not change, then no player will be worse off in the new situation.

Now we will provide two characterizations of the MC-value on subclasses of monotonic NTU-games. Attention will be restricted to the class $\widetilde{\Gamma}_m^N$ of monotonic NTU-games $V \in \Gamma_m^N$ satisfying $b(V) > 0$. This is a weak condition which means that every player has a positive marginal contribution to at least one coalition. Particularly, games with null players (i.e., players who contribute nothing to each coalition) are excluded.

We have the following characterization of the MC-value on the class $\widetilde{\Gamma}_m^N$ which is similar to Theorem 2.4.8.

Theorem 5.4.2 The MC-value is the unique value on $\widetilde{\Gamma}_m^N$ which satisfies

(i) (weak) Pareto optimality,

(ii) scale covariance,

(iii) b-symmetry.

Proof. From Proposition 5.4.1 it follows that the MC-value satisfies (i)-(iii). Let $F : \widetilde{\Gamma}_m^N \to \mathbf{R}^N$ satisfy the three properties, and let $V \in \widetilde{\Gamma}_m^N$. We show that $F(V) = MC(V)$.
Since $b(V) > 0$, the vector $\lambda \in \mathbf{R}_{++}^N$ with coordinates $\lambda_i := \frac{1}{b_i(V)}$ for all $i \in N$ is well-defined. Consider the game $\lambda * V$. Clearly, $\lambda * V \in \widetilde{\Gamma}_m^N$, and $b(\lambda * V) = \lambda * b(V) = e^N$. b-symmetry of F and the MC-value implies $F_i(\lambda * V) = F_j(\lambda * V)$ for all $i, j \in N$ and $MC_i(\lambda * V) = MC_j(\lambda * V)$ for all $i, j \in N$. From weak Pareto optimality of F and the MC-value it follows that $F(\lambda * V) = MC(\lambda * V)$. Scale covariance now yields $F(V) = MC(V)$. □

Finally, we present a characterization of the MC-value which is comparable to Theorem 2.4.9 (or 4.1.3). For this, we have to impose the extra condition that the set $V_0(N)$ is non-level (cf. (2.1)). Let $\widehat{\Gamma}_m^N$ denote the class of all $V \in \widetilde{\Gamma}_m^N$ for which $V_0(N)$ is non-level. Then we have the following analogue of Theorem 2.4.9.

Theorem 5.4.3 The MC-value is the unique value on $\widehat{\Gamma}_m^N$ which satisfies

(i) (weak) Pareto optimality,

(ii) scale covariance,

(iii) symmetry,

(iv) conditional monotonicity.

Proof. Let $F : \widehat{\Gamma}_m^N \to \mathbf{R}^N$ satisfy the properties (i)-(iv). From the proof of Theorem 5.4.2 it follows that it is sufficient to show that $F(V) = MC(V)$ for games $V \in \widehat{\Gamma}_m^N$ with $b(V) = e^N$. Let V be such a game. Since $b(V) > 0$, it follows that $MC(V) > 0$. Moreover, b-symmetry of the MC-value implies $MC_i(V) = MC_j(V)$ for all $i, j \in N$. Consider the following NTU-game W.[3]

[3]The construction of $W(N)$ is due to Henk Norde (private communication).

$$W(S) := \begin{cases} \{x \in \mathbf{R}^S \mid x \leq 0\}, & \text{if } S \neq N, \\ \{x \in V(N) \mid \sigma(x) \in V(N) \text{ for all } \sigma \in \Pi(N)\}, & \text{if } S = N. \end{cases}$$

Note that $W(N)$ is the largest symmetric subset of $V(N)$. Since $MC_i(V) = MC_j(V)$ for all $i, j \in N$, it follows that $MC(V) \in W(N)$, and because $MC(V) \in V(N) \setminus dom(V(N))$ it also follows that $MC(V) \in W(N) \setminus dom(W(N))$. Hence, $b(W) > 0$, and since $W(N)$ is symmetric, it easily follows that $b_i(W) = b_j(W)$ for all $i, j \in N$. So the origin 0, $b(V)$ and $b(W)$ are lying on one line.

Claim: $W \in \widehat{\Gamma}^N_m$.

Proof of the claim. It is sufficient to show that $W_0(N)$ is non-level. Let $x \in W_0(N) \setminus dom(W(N))$, and suppose there exists a $y \in W_0(N) \setminus dom(W(N))$ with $y \geq x, y \neq x$. Since $x, y \in W_0(N)$, we have $\sigma(x), \sigma(y) \in V_0(N)$ for all $\sigma \in \Pi(N)$. Moreover, $\sigma(y) \geq \sigma(x), \sigma(y) \neq \sigma(x)$ for all σ. Non-levelness of $V_0(N)$ implies that $\sigma(x) \in dom(V_0(N))$ for all σ. Hence, for each $\sigma \in \Pi(N)$ there exists an $\epsilon_\sigma \in \mathbf{R}_{++}$ such that $\mathcal{B}(\sigma(x), \epsilon_\sigma) := \{z \in \mathbf{R}^N \mid ||z - \sigma(x)|| < \epsilon_\sigma\} \subset V_0(N)$. Take $\epsilon := \min\{\epsilon_\sigma \mid \sigma \in \Pi(N)\}$. Then $\mathcal{B}(\sigma(x), \epsilon) \subset V_0(N)$ for all $\sigma \in \Pi(N)$, and since $\mathcal{B}(\sigma(x), \epsilon) = \sigma(\mathcal{B}(x, \epsilon))$, it follows that $\mathcal{B}(x, \epsilon) \subset W_0(N)$. Hence, $x \in dom(W_0(N))$, which yields a contradiction. So the claim is proved.

Symmetry and weak Pareto optimality of F and the MC-value imply $F(W) = MC(W)$.[4] Since $MC(V) \in W(N) \setminus dom(W(N))$, and 0, $b(V)$ and $b(W)$ are lying on one line, it follows that $MC(W) = MC(V)$. Further, conditional monotonicity of F yields that $F(V) \geq F(W) = MC(V)$. Since $MC(V) \in V_0(N) \setminus dom(V(N))$, non-levelness of $V_0(N)$ implies $F(V) = MC(V)$. □

We leave it to the reader to verify that in Theorems 5.4.2 and 5.4.3 the characterizing properties are independent. Theorem 5.4.3 illustrates that the characterization of the RKS-solution by Kalai and Smorodinsky (1975) (Theorem 2.3.2) can be extended in order to obtain a characterization of the MC-value. It remains an open problem, however, whether a characterization of the MC-value can be developed based on characterizations of the Shapley value (cf. Shapley (1953), Young (1985)).

We conclude this section with a comparison between the MC-value on the one hand and the compromise value and the egalitarian solution on the other hand.

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[4] Note that we only use an even weaker version of symmetry, which states that if all players in a game are symmetric, then they will all receive the same payoff.

Theorems 5.4.2 and 5.4.3 clearly reflect the similarity between the MC-value and the compromise value. By appropriately modifying the (bounds-) symmetry and monotonicity property which characterize the compromise value, one can obtain characterizing properties for the MC-value. Whereas Theorem 2.4.8 is almost identical to Theorem 5.4.2, there is a difference between Theorem 2.4.9 (or 4.1.3) and Theorem 5.4.3. In the latter we do not have to require a convexity condition on the sets $V(N)$, while in the first this seems a necessary condition. However, it is not difficult to show that by weakening the monotonicity property in Theorem 2.4.9 (or 4.1.3) in the spirit of conditional monotonicity, we can drop the convexity condition. It should be noted that Theorems 5.4.2 and 5.4.3 also hold if convexity conditions are imposed on the NTU-games (this is trivial for Theorem 5.4.2, and in Theorem 5.4.3 one only has to show that the NTU-game W, which is defined in the proof, is convex valued whenever V is convex valued).

Of course, also from a definitional viewpoint there is a clear similarity between the compromise value and the MC-value. Both values generalize the RKS-solution to NTU-games and both values can be considered as a compromise solution. An important difference between both solutions lies in the fact that for the compromise value the lower vector depends on the game, whereas for the MC-value this vector always equals the zero vector.

Another candidate for a comparison with the MC-value is the egalitarian solution introduced by Kalai and Samet (1985). Not only are both solution concepts defined and characterized for not necessarily convex valued NTU-games, but both extend the Shapley value to the class of NTU-games. Moreover, both solutions concepts yield a payoff vector which can be obtained by moving from 0 as a starting point in a certain direction in order to obtain a weak Pareto optimal outcome. However, for the egalitarian solution this direction is independent of the game under consideration, while for the MC-value the direction is determined by the game. (It should be said that the fixed direction for the egalitarian solution is computed in a recursive way with respect to the inclusion relation between coalitions.) As a consequence of the fixed direction the egalitarian solution does not satisfy the covariance property. Hence, the egalitarian solution depends on the utility representation of the preferences of the players.

The main difference in the domain of the characterizations of the egalitarian solution and the MC-value is that the egalitarian solution is characterized on a class of NTU-games for which no monotonicity and non-levelness condition are required. If we compare Theorem 5.4.3 with the characterization of the egalitarian solution as given in Kalai and Samet (1985), it is striking that in both characterizations a monotonicity property plays a crucial role. It should be remarked that the MC-value does not satisfy the monotonicity property which is used to characterize the egalitarian solution, and the latter does not satisfy the conditional monotonicity property introduced above.

Part II

Division problems with single-peaked preferences

Chapter 6

The uniform rule: an overview

We consider the problem of fairly distributing a non-negative amount of a perfectly divisible good among a finite set of agents who have single-peaked preferences, i.e., up to a certain amount an agent likes to consume more of the good, beyond this amount the opposite holds. Since in general the sum of the preferred consumptions will not be equal to the amount which has to be allocated among the agents, the problem of interest is how to achieve a fair division (free disposal of the good is assumed not to be allowed).

A possible interpretation of this model is the 'baby-sitting problem' as discussed in Section 1.2. Another interpretation given by Sprumont (1991) is that of a group of workers who have to contribute a certain quantity of labour in order to perform a certain job. If the total amount of work is fixed and each agent receives an hourly wage, then preferences over the participation levels are rather naturally single-peaked.

This problem has been studied extensively in the literature. Sprumont (1991) initiated the axiomatic analysis, by showing that there is a unique rule which satisfies Pareto optimality, strategy-proofness and either envy-freeness or anonymity. This rule, which he called the uniform rule, is an adaptation of the uniform rationing scheme introduced by Benassy (1982) in the fixed-price literature. The uniform rule allocates to each agent his preferred amount as long as it obeys a certain lower and upper bound which are the same for all agents and chosen such that feasibility is attained.

Ching (1994) shows that in Sprumont's characterization the anonymity property can be replaced by the weaker property of equal treatment of equals and he provides an alternative proof. Other characterizations of the uniform rule are given in Thomson

(1994a) using the well-known principles of consistency and converse consistency.
As a result of this extensive analysis, the uniform rule is now considered to be the most interesting rule for this type of problems. Thomson (1994a) formulated this as follows:

> *"On the basis of what we now know, the uniform rule can wholeheartedly be advocated as the best solution to the problem of fair division in economies with single-peaked preferences".*

This chapter presents an overview of the most important results in this framework. In Section 6.1 we formally describe the model and introduce some basic notions and solution concepts, among others we define the uniform rule. Section 6.2 discusses several properties of rules which are all satisfied by the uniform rule. Finally, in Section 6.3 we recall some of the characterizations of the uniform rule provided by Sprumont (1991), Ching (1994), and Thomson (1991, 1994a,b).

6.1 The model

Let $\overline{M}$ be some fixed positive number. Given any *preference relation* R over $[0, \overline{M}]$, which we assume is a complete and transitive binary relation, we denote $x\ R\ y$ if $(x, y) \in R$, $x\ P\ y$ if $x\ R\ y$ and not $y\ R\ x$, and $x\ I\ y$ if $x\ R\ y$ and $y\ R\ x$. R is called *single-peaked* if there exists a number $p(R) \in [0, \overline{M}]$ such that for all $x, y \in [0, \overline{M}]$, with $x < y \leq p(R)$ or $p(R) \leq y < x$, we have $y\ P\ x$. $p(R)$ is called the *peak* of the relation R. By $\mathcal{R}$ we denote the set of all single-peaked preferences over $[0, \overline{M}]$. The introduction of $\overline{M}$ is just for notational convenience: It allows us to define peaks as a function only of the preferences, i.e., independently of the amount to be divided. An alternative way would be to define preferences over $[0, \infty)$, but then monotonic increasing preferences would be excluded from the definition of single-peaked preferences unless we say that in this case the peak is infinity.
An *economy* is a tuple $E = < M, (R_i)_{i \in N} >$, where $0 \leq M \leq \overline{M}$, $N \subset \mathbb{N}$ is a finite, non-empty set of agents, and for each $i \in N$, $R_i \in \mathcal{R}$. Denote $p(E) := (p(R_i))_{i \in N}$. The class of all economies (with variable set of agents) is denoted by $\mathcal{E}$.
An economy represents the problem of allocating a positive amount of a perfectly divisible good, which cannot be disposed of, among a group of agents who have single-peaked preferences over $[0, \overline{M}]$.

Given an economy $E =< M, (R_i)_{i\in N} >$ the problem of interest is to determine an allocation in a fair way. An *allocation* for E is a vector $x \in \mathbb{R}_+^N$ such that $x(N) = M$. By $X^*(E)$ we denote the set of all allocations for E. An allocation $x \in X^*(E)$ is called *efficient* if there is no $y \in X^*(E)$ such that $y_i \; R_i \; x_i$ for all $i \in N$ and $y_i \; P_i \; x_i$ for some $i \in N$. $X(E)$ denotes the set of all efficient allocations for E.

Sprumont (1991) showed that an allocation for an economy is efficient if and only if there are no two agents such that one gets more than his peak and the other gets less than his peak. This means that an allocation is efficient if and only if all agents are on the "same side" of their peaks. Formally, for an economy $E =< M, (R_i)_{i\in N} >\in \mathcal{E}$ and $x \in X^*(E)$,

$$x \in X(E) \Leftrightarrow \begin{cases} x \le p(E) & \text{if } M \le \sum_{i\in N} p(R_i) \\ x \ge p(E) & \text{if } M \ge \sum_{i\in N} p(R_i). \end{cases}$$

A *rule* ϕ is a function which assigns to each economy $E \in \mathcal{E}$ an allocation $\phi(E) \in X^*(E)$, which can be interpreted as a recommendation for the economy E. Simple examples of rules are the *egalitarian rule*, which distributes the amount equally among the agents, and the *proportional rule* which allocates the amount in proportion to the peaks of the agents. Other simple examples are the *equal distance rule*, which selects an efficient allocation such that the distance between the allocated amount and the peak is the same for each agent (as long as it does not conflict feasibility) (see Thomson (1991)), and the *queueing rule* (see Sprumont (1991)). To apply the queueing rule the agents are numbered $1, \ldots, |N|$ in a random way and, in case there is too much of the good, every agent $1 \le i \le |N| - 1$ is assigned his peak. The remaining part is allocated to agent $|N|$. In case the amount of the good is less than the sum of the peaks, we start giving (in the order $1, \ldots, |N|$) to the agents the amount that corresponds to their peak. If at some point there is not sufficient left for some agent i, he obtains the remaining part. All remaining agents obtain 0 .

A rule which plays a central role in the literature of economies with single-peaked preferences is the uniform rule. The *uniform rule*, U, is defined as follows. Let $E =< M, (R_i)_{i\in N} >\in \mathcal{E}$ be an economy and let $i \in N$. Then

$$U_i(E) := \begin{cases} \min\{p(R_i), \lambda\} & \text{if } M \le \sum_{i\in N} p(R_i) \\ \max\{p(R_i), \lambda\} & \text{if } M \ge \sum_{i\in N} p(R_i), \end{cases}$$

where λ is such that $U(E) \in X^*(E)$.

For the case in which there is too little to divide, i.e., $M < \sum_{i\in N} p(R_i)$, the uniform

rule chooses appropriately an amount λ and allocates it to every agent with peak above this amount while all other agents obtain their peak. Here, appropriately means that the resulting division is indeed an allocation.
Note that the uniform rule as well as the other rules discussed above only depend on the amount to be divided and the peaks of the players. So from a normative perspective these rules may be criticized. From an informational viewpoint however, these so-called 'peak-only' rules are very appealing because it is not necessary to know the exact preference structure of the agents.

Example 6.1.1 To illustrate the uniform rule let us reconsider the baby-sitting problem from Section 1.2 in which there are four people who have to baby-sit for sixteen hours. The preferred points of the four persons, numbered 1, 2, 3, and 4, are 1, 3, 4, 5 (hours), respectively. In this case the sum of the peaks is less than the amount to be divided, which implies that in order to obtain an efficient outcome every agent has to baby-sit at least his preferred time. The uniform rule yields the outcome $(3\frac{1}{2}, 3\frac{1}{2}, 4, 5)$, which one might say favours persons 3 and 4 above the two others.

6.2 Properties of the uniform rule

Although Example 6.1.1 might suggest that the uniform rule cannot be considered as a 'fair' allocation rule, one of the reasons why the uniform rule is interesting, is that it is the only rule which satisfies many desirable properties. This section summarizes several properties of the uniform rule.
Several standard properties of the uniform rule are listed in Proposition 6.2.1.

Proposition 6.2.1 The uniform rule satisfies the following properties.

(i) *Pareto optimality*: $U(E) \in X(E)$ for all $E \in \mathcal{E}$.

(ii) *Anonymity*: $U_i(E^\sigma) = U_{\sigma(i)}(E)$ for all $E =< M, (R_i)_{i\in N} >\in \mathcal{E}$, all $i \in N$, and all $\sigma \in \Pi(N)$, where $E^\sigma :=< M, (R_{\sigma(i)})_{i\in N} >$.

(iii) *Equal treatment (of equals)*: $U_i(E)\ I_i\ U_j(E)$ for all $E =< M, (R_i)_{i\in N} >\in \mathcal{E}$ and all $i, j \in N$ with $R_i = R_j$.

(iv) *Peak only*: $U(E) = U(E')$ for all $E =< M, (R_i)_{i\in N} >$, $E' =< M, (R'_i)_{i\in N} >\in \mathcal{E}$ with $p(E) = p(E')$.

(v) *M-continuity*: U is continuous with respect to the amount to be divided.

(vi) *Envy-freeness*: $U_i(E)\ R_i\ U_j(E)$ for all $E \in \mathcal{E}$ and all $i, j \in N$.

(vii) *Strategy-proofness*: $U_i(E)\ R_i\ U_i(E_i')$ for all $E =< M, (R_i)_{i \in N} >$, all $i \in N$, and all $R_i' \in \mathcal{R}$, where $E_i' :=< M, (R_i', (R_j)_{j \in N \setminus \{i\}}) >$.

(viii) *Individually rational from equal division*: $U_i(E)\ R_i\ \frac{M}{|N|}$ for all $i \in N$ and all $E \in \mathcal{E}$.

Properties (i)-(iii) and (v) are standard and need no further explanation. Peak only requires from a rule to take into consideration only the peaks of the preference profile when dividing a certain amount M. Peak only is a natural property for rules which satisfy Pareto optimality. To see this recall that a Pareto optimal rule selects allocations which are characterized by the fact that either all agents get more than their peaks or all agents get less than their peaks. Once restricted to the relevant side of the peak all preferences with this peak are identical. Hence, the peak contains all the 'relevant' information. Envy-freeness is also well-known and was first introduced by Foley (1967). It states that no agent prefers someone else's amount to his own share. Strategy-proofness states that if the outcome is applied on the basis of declared preferences, it is a (weakly) dominant strategy for each player to reveal his true preferences. This is also a desirable property, since it implies that every agent only needs to know his own preferences to determine his best choice. Property (viii) implies that every agent prefers the amount allocated to him to the equal division outcome. This property is meaningful in fair division problems since often the equal division outcome is considered as a starting point which represents the natural right or ownership of each agent.

This list of properties of the uniform rule is certainly not complete. Other properties satisfied by the uniform rule are several monotonicity properties (among others one-sided resource monotonicity and one-sided population monotonicity). We will not discuss these properties here in detail, but refer the reader to Thomson (1991, 1994b).

Thomson (1994a) investigates several consistency properties of the uniform rule. In order to introduce them, we need the following notation.

Let $E =< M, (R_i)_{i \in N} > \in \mathcal{E}$ be an economy, $x \in X^*(E)$, and $S \subset N$, $S \neq \emptyset$. The *reduced economy* with respect to S and x is

$E^{S,x} :=< x(S), (R_i)_{i\in S} >$.

Remark 6.2.2 Note that $E^{S,x} \in \mathcal{E}$. Further, if $\emptyset \neq T \subset S$, then $E^{T,x} = [E^{S,x}]^{T,x_S}$.

A rule ϕ is called *consistent* if for all economies $E =< M, (R_i)_{i\in N} >\in \mathcal{E}$ and all $S \subset N$, $S \neq \emptyset$ we have that $x = \phi(E)$ implies $x_S = \phi(E^{S,x})$.

Roughly speaking, consistency of a rule means that, if a subgroup of agents would decide to pool their parts of the allocation prescribed by the rule and apply the same rule to redistribute this total, then the agents in that group would end up each with the same amount as before. Thomson (1994a) proved that the uniform rule is consistent.

A rule ϕ is called *converse consistent* if for all economies $E \in \mathcal{E}$ we have that $x \in X^*(E)$ and $x_S = \phi(E^{S,x})$ for all $S \subset N$ with $|S| = 2$, implies $x = \phi(E)$.

Converse consistency means that, given a certain allocation x for an economy, if the restriction of x is recommended for every reduced economy with respect to a subgroup of two agents and x, then the allocation x is recommended in the large economy. Thomson (1994a) noted that the uniform rule satisfies this property. Converse consistency of the uniform rule also follows from the next lemma which establishes a relation between consistency and converse consistency for this model. Before we formulate this relation we need the following additional property for rules.

M-Monotonicity: A rule ϕ is M-monotonic if for all $E =< M, (R_i)_{i\in N} >\in \mathcal{E}$, and $E' =< M', (R_i)_{i\in N} >\in \mathcal{E}$, with $M \leq M'$, we have $\phi(E) \leq \phi(E')$.

M-monotonicity is different from the 1-sided resource monotonicity introduced in Thomson (1994b). However, if Pareto optimality is imposed both properties are equivalent. Clearly, the uniform rule satisfies M-monotonicity.
The following lemma illustrates that for M-monotonic rules converse consistency is implied by consistency.

Lemma 6.2.3 Let ϕ be an M-monotonic rule. Then ϕ is consistent if and only if (i) for every economy $E \in \mathcal{E}$ there exists an $x \in X^*(E)$ such that $x_S = \phi(E^{S,x})$ for all $S \subset N$ with $|S| = 2$, and (ii) ϕ is converse consistent.

Proof. ($\Rightarrow$) Let $E \in \mathcal{E}$. Take $x := \phi(E)$. Consistency of ϕ yields that $x_S = \phi(E^{S,x})$ for all $S \subset N$ with $|S| = 2$. In order to prove that ϕ is converse consistent, it suffices to show that there is no allocation $y \in X^*(E)$, $y \neq x$, such that $y_S = \phi(E^{S,y})$ for all $S \subset N$ with $|S| = 2$. Suppose that there exists such a y . Since $x(N) = y(N) = M$, it follows that there are $i, j \in N$ such that $x_i < y_i$ and $x_j > y_j$. Take $S := \{i, j\}$. Without loss of generality we assume that $x(S) \geq y(S)$. M-Monotonicity of ϕ yields that $\phi(E^{S,x}) \geq \phi(E^{S,y})$. Hence, $x_S \geq y_S$, which yields a contradiction.
($\Leftarrow$) Let $E \in \mathcal{E}$. Let $\emptyset \neq T \subset N$ and $x = \phi(E)$. We have to prove that $x_T = \phi(E^{T,x})$. By assumption there exists a $y \in X^*(E)$ such that $y_S = \phi(E^{S,y})$ for all $S \subset N$ with $|S| = 2$. Converse consistency of ϕ yields that $y = \phi(E) = x$. Hence, $x_S = \phi(E^{S,x})$ for all $S \subset N$ with $|S| = 2$. By Remark 6.2.2, $x_S = \phi([E^{T,x}]^{S,x_T})$ for all $S \subset T$, with $|S| = 2$. Clearly, $x_T \in X^*(E^{T,x})$. Hence, converse consistency of ϕ yields $x_T = \phi(E^{T,x})$. □

6.3 Characterizations of the uniform rule

In this section we recall several characterizations of the uniform rule, which are based on the properties discussed in the previous section.
The first characterizations of the uniform rule are provided by Sprumont (1991) who proved the following two results.

Theorem 6.3.1 The uniform rule is the unique rule which satisfies Pareto optimality, strategy-proofness, and anonymity.

If anonymity is replaced by envy-freeness, another characterization of the uniform rule is obtained.

Theorem 6.3.2 The uniform rule is the unique rule which satisfies Pareto optimality, strategy-proofness, and envy-freeness.

Theorems 6.3.1 and 6.3.2 are strengthened in Ching (1994), who showed that in these characterizations anonymity and envy-freeness can be replaced by the weaker equal treatment property. Moreover, Ching provides an alternative proof of this result.
Thomson (1994a) provides several characterizations of the uniform rule using consistency and converse consistency. We will not discuss all these characterizations in detail, but mention two of his results.

Theorem 6.3.3 The uniform rule is the unique rule which satisfies Pareto optimality, envy-freeness, consistency, and M-continuity.

It turns out that in this characterization envy-freeness can be replaced by individually rationality from equal division. Another characterization using this property together with converse consistency is given in the following theorem.

Theorem 6.3.4 The uniform rule is the unique rule which satisfies Pareto optimality, anonymity, converse consistency, and individually rationality from equal division.

Other characterizations of the uniform rule that use monotonicity properties (population monotonicity and resource monotonicity) are given in Thomson (1991, 1994b).

In the following chapter we establish a relation between the uniform rule and two well-known solutions for bargaining problems. These relations give rise to two new characterizations of the uniform rule.

Chapter 7

Alternative characterizations of the uniform rule

This chapter, which is based on Otten, Peters and Volij (1994), provides two new characterizations of the uniform rule, both of which are inspired by the characterizations of two different bargaining solutions.

In Section 7.2 we associate with each economy an auxiliary bargaining problem in such a way that the set of Pareto optimal allocations of the bargaining problem coincides with the set of efficient divisions in the original economy. Next we show that for each economy the division recommended by the uniform rule, coincides with the allocation recommended by both the Nash and the lexicographic egalitarian bargaining solutions to the associated bargaining problem. The proofs of these statements are interesting because they use the principles of consistency and converse consistency in different contexts, namely in the context of bargaining problems on the one hand, and of the allocation of a commodity among agents with single-peaked preferences on the other hand. Moreover, they illustrate that consistency and converse consistency, which have been employed in several characterizations of game theoretic solution concepts (cf. Sobolev (1975), Peleg (1985, 1986), Lensberg (1988), Hart and Mas-Colell (1989), Peleg and Tijs (1992), and Chapter 3), can be helpful for other purposes as well.

Both statements suggest that the uniform rule might be characterized by means of some suitably adapted set of properties that characterize the bargaining solutions mentioned above. Section 7.3 discusses properties of the uniform rule reminiscent of the independence of irrelevant alternatives (IIA) property used by Nash (1950) to characterize the Nash bargaining solution and the restricted monotonicity property introduced by Chun and Peters (1988) to characterize the lexicographic egalitarian bargaining solution.

Section 7.4 presents two characterizations of the uniform rule which are based on the properties introduced in Section 7.3.

Finally, Section 7.5 translates our results on the uniform rule to a classical cost-sharing model and obtains a characterization of the head tax mechanism.

Since our results in Sections 7.2-7.4 are proved and inspired by well-known results from cooperative bargaining theory, we first recall some basic notions of bargaining theory in Section 7.1.

7.1 Bargaining problems

Recall that $N \subset \mathbb{N}$ denotes a finite, non-empty set of agents. A *bargaining problem* for N is a subset B of $\mathbf{R}_+^N$ which satisfies the following properties:

(i) B is compact and convex.

(ii) There exists a $y \in B$ with $y > 0$.

(iii) B is comprehensive, i.e., if $x \in B$, and $y \in \mathbf{R}_+^N$, with $y \leq x$, then $y \in B$.[1]

Let $\mathcal{B}$ denote the set of all bargaining problems (with variable set of agents).[2]

A *(bargaining) solution* is a function $\mathcal{F}$ which assigns to each $B \in \mathcal{B}$ an element $\mathcal{F}(B)$ of B.

A prominent solution is the Nash bargaining solution introduced by Nash (1950). Let $B \in \mathcal{B}$ be a bargaining problem for N. The *Nash bargaining solution* is defined by

$$\mathcal{N}(B) := \text{argmax}\{\prod_{i \in N} x_i \mid x \in B\}.$$

Another bargaining solution is the lexicographic egalitarian solution. To define it we need some notation.

Let $\alpha : \mathbf{R}^N \to \mathbf{R}^N$ be a function such that for each $x \in \mathbf{R}^N$ the vector $\alpha(x)$ is a reordering of the coordinates of x in a non-decreasing order. So if $i, j \in N$ with $i < j$,

[1] Note that attention is restricted to $\mathbf{R}_+^N$ (cf. Section 2.3).

[2] In Section 2.3 we defined a bargaining problem by a set B and a disagreement outcome $d \in B$. For our analysis in this chapter the disagreement outcome does not play an explicit role: The reader may think of the disagreement point as being $d = 0$.

then $\alpha_i(x) \leq \alpha_j(x)$. The *lexicographic maximin ordering* $\geq^{lm}$ on $\mathbf{R}^N$ is defined by $x \geq^{lm} y$ if $\alpha(x) \geq^{l} \alpha(y)$, where $\geq^{l}$ denotes the lexicographic order on $\mathbf{R}^N$ (cf. Section 2.1).

The *lexicographic egalitarian solution*, $\mathcal{L} : \mathcal{B} \to \mathbf{R}^N$ assigns to each bargaining problem $B \in \mathcal{B}$ the unique point which is maximal with respect to the lexicographic maximin ordering $\geq^{lm}$.

It is well-known that both $\mathcal{N}$ and $\mathcal{L}$ satisfy the properties listed below.

> A solution $\mathcal{F}$ is *Pareto optimal* if for all $B \in \mathcal{B}$ and all $y \in B$ we have, if $y \geq \mathcal{F}(B)$, then $y = \mathcal{F}(B)$.
>
> A solution $\mathcal{F}$ satisfies *strict individual rationality* if $\mathcal{F}(B) > 0$ for all $B \in \mathcal{B}$.

Lensberg (1988) and Thomson and Lensberg (1989) characterized the lexicographic egalitarian solution and the Nash bargaining solution respectively, by means of a consistency property. In order to introduce it we need the following definition.
Let $B \in \mathcal{B}$ be a bargaining problem for N, let $x \in B$, and let $S \subset N$, $S \neq \emptyset$. The *reduced bargaining problem* with respect to S and x is

$$B^{S,x} := \{y_S \in \mathbf{R}_+^S \mid (y_S, x_{N\setminus S}) \in B\}.$$

Note that not necessarily for every $x \in B$ it holds that $B^{S,x} \in \mathcal{B}$. However, if $x = \mathcal{N}(B)$ or if $x = \mathcal{L}(B)$, then $B^{S,x} \in \mathcal{B}$. This is a consequence of the fact that both $\mathcal{N}$ and $\mathcal{L}$ satisfy strict individual rationality.
The consistency property is now defined as follows.

> A solution $\mathcal{F}$ is *consistent* if for all bargaining problems $B \in \mathcal{B}$ for N, and all $S \subset N$, $S \neq \emptyset$ with $B^{S,x} \in \mathcal{B}$ where $x = \mathcal{F}(B)$ we have that $x_S = \mathcal{F}(B^{S,x})$.

For the results in the next section we make use of the fact that both $\mathcal{N}$ and $\mathcal{L}$ satisfy the consistency property. The results in Section 7.3 are based on the characterization of $\mathcal{N}$ by Nash (1950) (cf. Section 2.3), and the characterization of $\mathcal{L}$ by Chun and Peters (1988) in which a restricted monotonicity property plays a central role.

7.2 New formulations of the uniform rule

We start with some notation. Let $E =< M, (R_i)_{i\in N} >\in \mathcal{E}$ be an economy. Let $\rho(E)$ denote the set of agents $i \in N$ for which there is an $x \in X(E)$ such that $x_i > 0$. Note that $\rho(E) \neq \emptyset$ if and only if $M > 0$. If one is interested in Pareto optimal rules, it is clear that the problem is, how to divide the total amount M among the agents in $\rho(E)$, for all efficient allocations give zero to the agents not in $\rho(E)$. In other words, the set of agents which are *relevant* for economy E is $\rho(E)$.

The following two theorems present alternative formulations of the uniform rule and reflect the similarity between the uniform rule on the one hand and the Nash and lexicographic egalitarian solution on the other hand.

Theorem 7.2.1 Let $E =< M, (R_i)_{i \in N} > \in \mathcal{E}$ be an economy. Then $U(E)$ is the unique element of $\text{argmax}\{\prod_{i \in \rho(E)} x_i \mid x \in X(E)\}$ if $\rho(E) \neq \emptyset$, and $U(E) = 0$, otherwise.

Theorem 7.2.2 Let $E =< M, (R_i)_{i \in N} > \in \mathcal{E}$ be an economy. Then $U(E)$ is the unique efficient allocation for E which is maximal with respect to $\geq^{lm}$.

Instead of giving a direct proof of both theorems, we will give an indirect one based on some properties of the uniform rule and the consistency property of both the Nash solution and the lexicographic egalitarian solution.

Proof of Theorems 7.2.1 and 7.2.2.
Clearly, both theorems hold if the economy consists of only one agent or if $M = 0$. So from now on attention is restricted to economies with at least two agents and $M > 0$. For any such economy $E =< M, (R_i)_{i \in N} >$ we define

$$B(E) := comp(X(E)) = \{x \in \mathbf{R}^N_+ \mid x \leq y \text{ for some } y \in X(E)\} \text{ (see Figure 7.1.)}$$

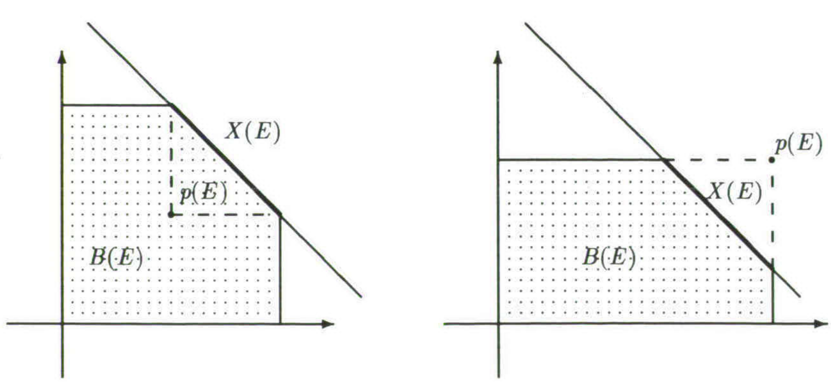

Figure 7.1. The set B(E) in case E is an economy with two agents.

Case 1: All agents are relevant.
Let $E =< M, (R_i)_{i \in N} >$ be an economy with $\rho(E) = N$ and $|N| \geq 2$. Since, $\rho(E) = N$, and $X(E)$ is a convex set, there exists a point $y \in X(E)$ with $y > 0$. Hence, $B(E)$ is a bargaining problem. $B(E)$ is called the ***bargaining problem associated with*** E. (It should be noted that $B(E)$ represents a set of physical allocations, whereas a bargaining problem in the usual sense represents a set of utility n-tuples.)

The following lemma shows that the operation of reducing an economy commutes with the operation of reducing an associated bargaining problem. It also implies that, within this context, the consistency requirements for bargaining problems and economies coincide.

Lemma 7.2.3 Let $E =< M, (R_i)_{i \in N} >$ be an economy with $|N| \geq 2$ and $\rho(E) = N$. Let $S \subset N$, $S \neq \emptyset$, and $x \in X(E)$. Then

$$B(E^{S,x}) = B^{S,x}(E).$$

Proof. We only prove the statement for $\sum_{i \in N} p(R_i) \leq M$. The other case is easier. Since $x \in X(E)$, it follows that $\sum_{i \in S} p(R_i) \leq x(S)$. Hence,

$$X(E^{S,x}) = \{y \in \mathbf{R}_+^S \mid y(S) = x(S), y_i \geq p(R_i) \ \forall i \in S\}.$$

Let $y \in B(E^{S,x}) = comp(X(E^{S,x}))$. Then there exists a $z \in X(E^{S,x})$ with $z \geq y$. This means that $z(S) = x(S)$, and $z_i \geq p(R_i)$ for all $i \in S$, which implies $(z, x_{N \setminus S}) \in X(E) \subset B(E)$. Hence, by definition of the reduced bargaining problem, it follows that $z \in B^{S,x}(E)$. Since $B^{S,x}(E)$ is comprehensive, we have $y \in B^{S,x}(E)$.
Now take $y \in B^{S,x}(E) = \{y \in \mathbf{R}_+^S \mid (y, x_{N \setminus S}) \in comp(X(E))\}$. Then there exists a $t \in X(E)$ with $t \geq (y_S, x_{N \setminus S})$ and $t_i \geq p(R_i)$ for all $i \in S$. Since $t_{N \setminus S} \geq x_{N \setminus S}$ and $t(N) = x(N)$, it follows that $t(S) \leq x(S)$. Hence, $t_S \in comp(\{z \in \mathbf{R}_+^S \mid z(S) = x(S),\ z_i \geq p(R_i) \text{ for all } i \in S\}) = B(E^{S,x})$. Since $t_S \geq y_S$, comprehensiveness of $B(E^{S,x})$ implies that $y_S \in B(E^{S,x})$. □

In order to prove Theorems 7.2.1 and 7.2.2 for Case 1 it is sufficient to show that

$$U(E) = \mathcal{N}(B(E)) = \mathcal{L}(B(E)). \tag{7.1}$$

First note that in case $|N| = 2$, it is clear that $U(E) = \mathcal{L}(B(E))$. Furthermore, it is also straightforward to show that $\mathcal{N}(B(E)) = \mathcal{L}(B(E))$.

Hence, it remains to show that (7.1) holds if $|N| > 2$. This will follow from Lemma 7.2.4 below.
Let $\mathcal{E}' \subset \mathcal{E}$ be the family of economies $E = < M, (R_i)_{i \in N} >$ with $\rho(E) = N$.

Lemma 7.2.4 Let $\mathcal{F}$ be a bargaining solution which satisfies Pareto optimality, strict individual rationality, and consistency. If $\mathcal{F}(B(E)) = U(E)$ for all $E = < M, (R_i)_{i \in N} > \in \mathcal{E}'$ with $|N| = 2$, then $\mathcal{F}(B(E)) = U(E)$ for all $E \in \mathcal{E}'$.

Proof. Let $E \in \mathcal{E}'$ be an economy and define $x := \mathcal{F}(B(E))$. From strict individual rationality we know that $x > 0$ and therefore, $B^{S,x}(E) \in \mathcal{B}$. Moreover, by consistency of $\mathcal{F}$

$$x_S = \mathcal{F}(B^{S,x}(E)) \quad \text{for all } S \subset N,\ |S| = 2.$$

Furthermore, from Pareto optimality of $\mathcal{F}$ and the definition of $B(E)$, it follows that $x \in X(E)$. So by Lemma 7.2.3, $B(E^{S,x}) = B^{S,x}(E)$. Hence,

$$x_S = \mathcal{F}(B(E^{S,x})) \quad \text{for all } S \subset N,\ |S| = 2.$$

Since $B(E^{S,x}) \in \mathcal{B}$, it follows that there exists an $y \in X(E^{S,x})$ with $y > 0$. So $\rho(E^{S,x}) = S$ for all $S \subset N$, $|S| = 2$. Hence, by assumption

$$x_S = U(E^{S,x}) \quad \text{for all } S \subset N,\ |S| = 2.$$

Converse consistency of the uniform rule now yields

$$x = U(E).$$

□

Since both $\mathcal{N}$ and $\mathcal{L}$ are consistent, strict individually rational, and Pareto optimal bargaining solutions, which satisfy (7.1) in case E is an economy with two agents, it immediately follows from Lemma 7.2.4 that (7.1) holds for all $E \in \mathcal{E}'$. This finishes the proof of Case 1.

Case 2: Not all agents are relevant.
To complete the proof of Theorems 7.2.1 and 7.2.2 we consider an economy $E = < M, (R_i)_{i \in N} >$ with $\rho(E) \neq N$.

Let $x := U(E)$ and $S := \rho(E)$. $S \neq \emptyset$ since $M > 0$. Pareto optimality of U implies that $x_{N \setminus S} = 0_{N \setminus S}$. Consistency of U implies that $x_S = U(E^{S,x})$. Clearly, $\rho(E^{S,x}) = S$.

So by Case 1, we have $x_S = \text{argmax}\{\prod_{i\in S} y_i \mid y \in X(E^{S,x})\}$, and moreover, we have that x_S is maximal with respect to $\geq^{lm}$ in $X(E^{S,x})$. Since $X(E) = X(E^{S,x}) \times \{0_{N\setminus S}\}$, it follows that $U(E) = (x_S, 0_{N\setminus S}) = \text{argmax}\{\prod_{i\in S} y_i \mid y \in X(E)\}$, and that x is maximal with respect to $\geq^{lm}$ in $X(E)$. □

A similar kind of proof can be found in Aumann and Maschler (1985), who showed that a bankruptcy rule, the contested garment consistent rule, can be defined as the nucleolus of an appropriately chosen TU-game. Theorem 7.2.1 can be seen as a generalization of Dagan and Volij (1993) who showed that the constrained equal award rule for bankruptcy problems corresponds to the Nash bargaining solution of an appropriately chosen bargaining problem.

Remark 7.2.5 Another way to obtain the uniform rule, which was formulated by Thomson, is the following: To an economy E the uniform rule assigns the unique efficient allocation at which the difference between the smallest and largest consumption is the smallest. Formally, $U(E) = \text{argmin}\{\max_{i\in N}\{x_i\} - \min_{i\in N}\{x_i\} \mid x \in X(E)\}$.

7.3 Additional properties of the uniform rule

It is clear from the previous section that, at least formally, there is a relation between the uniform rule on the one hand, and the Nash and the lexicographic egalitarian bargaining solutions on the other hand. This suggests that the uniform rule might be characterized by means of a suitable adaptation of some properties that characterize these bargaining solutions. Before we go into characterizations of the uniform rule, we discuss in this section some properties inspired by the results of Section 7.2.
The following property, though different, is reminiscent of the one used by Nash (1950) in his characterization of the Nash bargaining solution.
Let ϕ be a rule.

> *Independence of irrelevant alternatives (IIA):* ϕ satisfies IIA if for all economies $E =< M, (R_i)_{i\in N} >, E' =< M, (R'_i)_{i\in N} >\in \mathcal{E}$ with $X(E) \subset X(E')$ we have that $\phi(E') \in X(E)$ implies $\phi(E) = \phi(E')$.

The IIA axiom makes sense only if ϕ is Pareto optimal. The idea behind this axiom is the following. If some efficient allocations which were not selected by the rule become inefficient, then this should not result in a change of the recommended outcome if this outcome is still efficient.

Lemma 7.3.1 The uniform rule satisfies IIA.

The proof follows directly from Theorem 7.2.1.

For our results we only need a weaker version of IIA which requires independence of irrelevant alternatives only in cases where in both economies either there is too much to divide or there is too little to divide.

> *One-sided independence of irrelevant alternatives:* ϕ satisfies one-sided IIA if for all $E =< M, (R_i)_{i \in N} >$, $E' =< M, (R_i')_{i \in N} > \in \mathcal{E}$ with $X(E) \subset X(E')$ such that $\max\{\sum_{i \in N} p(R_i), \sum_{i \in N} p(R_i')\} < M$ or $\min\{\sum_{i \in N} p(R_i), \sum_{i \in N} p(R_i')\} \geq M$ the following condition holds: if $\phi(E') \in X(E)$, then $\phi(E) = \phi(E')$.

Clearly, one-sided IIA is a weakening of IIA, so the uniform rule satisfies one-sided IIA.

Before we discuss a monotonicity property for the uniform rule which is based on the restricted monotonicity property introduced in Chun and Peters (1988), we first consider the following property.

> *Monotonicity:* ϕ satisfies monotonicity if for all economies $E =< M, (R_i)_{i \in N} >$ and $E' =< M', (R_i')_{i \in N} >$, such that for each $x \in X(E)$ there exists an $x' \in X(E')$ with $x_i' \; R_i' \; x_i$ for all $i \in N$ we have $\phi_i(E') \; R_i' \; \phi_i(E)$ for all $i \in N$.

Monotonicity states that if for every efficient allocation x in E we can find an efficient allocation x' in E' such that x' is weakly preferred to x by all agents in E', then the same must be true for the recommendations $\phi(E')$ and $\phi(E)$, namely $\phi(E')$ must be weakly preferred to $\phi(E)$ by all agents in E'. This property is similar in spirit to the monotonicity property in bargaining theory, and, like in bargaining theory, monotonicity is incompatible with Pareto optimality, as the following lemma shows.

Lemma 7.3.2 There is no Pareto optimal rule ϕ that satisfies monotonicity.

Proof. Let E, E' and E'' be three 2-agent economies in which there are 3 units to be divided. The peaks of the preference relations are $p = (1,2)$, $p' = (2,1)$, and $p'' = (3,3)$, respectively. By Pareto optimality of ϕ we have that $\phi(E) = (1,2)$. It is clear that $X(E) \subset X(E'')$ so E and E'' trivially satisfy the condition in the monotonicity property. Hence by monotonicity, we must have $\phi(E'') = (1,2)$. A similar argument shows that $\phi(E'') = (2,1)$, which is a contradiction. □

Lemma 7.3.2 shows that if we want to keep Pareto optimality, we must, as in bargaining theory, weaken the monotonicity requirement. We are going to weaken monotonicity in two different ways. First, we are going to allow for non-monotonicity only if one of the agents that got his peak in the economy E, strictly prefers the recommendation for the economy E'. Second, we are going to require this restricted form of monotonicity only when comparing some very specific economies.

> *One-sided restricted monotonicity:* ϕ satisfies one-sided restricted monotonicity if for all economies $E =< M, (R_i)_{i \in N} >, E' =< M, (R'_i)_{i \in N} > \in \mathcal{E}$, satisfying $X(E) \subset X(E')$ and either $\max\{\sum_{i \in N} p(R_i), \ \sum_{i \in N} p(R'_i)\} < M$ or $\min\{\sum_{i \in N} p(R_i), \ \sum_{i \in N} p(R'_i)\} \geq M$ the following condition holds: If $\phi_i(E) \ R'_i \ \phi_i(E')$ for all $i \in N$ such that $\phi_i(E) = p(R_i)$, then $\phi_i(E') \ R'_i \ \phi_i(E)$ for all $i \in N$.

In order to understand this axiom, note that $\phi_i(E) = p(R_i)$ means that it is physically impossible to make agent i better off in economy E. In this case we say that i's peak is binding at $\phi(E)$. One-sided restricted monotonicity says that given two economies E and E' satisfying the conditions in the definition of this property, if ϕ does not behave monotonically, i.e., there is some agent in E' who strictly prefers $\phi(E)$ to $\phi(E')$, then there must be some other agent in E', whose peak was binding at $\phi(E)$, who strictly prefers $\phi(E')$ to $\phi(E)$. In other words, if the peaks of the agents' preferences change in the same direction, then no agent's award should follow the opposite direction unless there is an agent whose peak was binding in the original situation and whose award followed the direction of his peak in the transition to the new situation. The motivation for this axiom is the same as the one for the restricted monotonicity satisfied by the lexicographic egalitarian bargaining solution (Chun and Peters (1988)): In some situations an agent may benefit from the fact that it is physically impossible to make other agents better off. If this impossibility disappears due to the fact that the peaks change, it may be bad news for those who benefitted from the previous situation, i.e., their awards may go farther away from their peaks. It is only this kind of non-monotonic behavior that is allowed by the restricted monotonicity axiom.

Lemma 7.3.3 The uniform rule satisfies one-sided restricted monotonicity.

Proof. Let $E =< M, (R_i)_{i \in N} >$ and $E' =< M, (R'_i)_{i \in N} > \in \mathcal{E}$ be two economies satisfying $\max\{\sum_{i \in N} p(R_i), \ \sum_{i \in N} p(R'_i)\} < M$ (the other case is similar) and assume

$X(E) \subset X(E')$. Then $p(E) \geq p(E')$. For all $i \in N$, let $U_i(E) = \max\{p(R_i), \lambda\}$ and $U_i(E') = \max\{p(R'_i), \lambda'\}$. Define $K := \{i \in N \mid U_i(E) > p(R_i)\}$ and assume $U_i(E)\ R'_i\ U_i(E')$, for all $i \in N \setminus K$, i.e.,

$$U_i(E) \leq U_i(E'), \quad \text{for all } i \in N \setminus K. \tag{7.2}$$

We need to show that $U_i(E')\ R'_i\ U_i(E)$ for all $i \in N$. Since

$$M = \sum_{i \in N} \max\{p(R'_i), \lambda'\} = \sum_{i \in N} \max\{p(R_i), \lambda\} \geq \sum_{i \in N} \max\{p(R'_i), \lambda\},$$

it follows that $\lambda' \geq \lambda$.

Take $i \in K$. It follows from the definition of K that $p(R'_i) \leq p(R_i) \leq \lambda \leq \lambda'$. Hence,

$$U_i(E) \leq U_i(E'). \tag{7.3}$$

This together with assumption (7.2) implies that (7.3) holds for all $i \in N$. But since $\sum_{i \in N} U_i(E) = \sum_{i \in N} U_i(E')$, we have $U_i(E) = U_i(E')$ for all $i \in N$, which in turn implies that $U_i(E')\ R'_i\ U_i(E)$ for all $i \in N$. □

The following lemma shows that there is a relation between the one-sided monotonicity and one-sided IIA.

Lemma 7.3.4 Let ϕ be a Pareto optimal rule. If ϕ satisfies one-sided restricted monotonicity, then ϕ satisfies one-sided IIA.

Proof. Let $E =< M, (R_i)_{i \in N} >$ and $E' =< M, (R'_i)_{i \in N} > \in \mathcal{E}$ be two economies satisfying $X(E) \subset X(E')$. We distinguish two cases.

Case 1: $\min\{\sum_{i \in N} p(R_i), \sum_{i \in N} p(R'_i)\} \geq M$.

Assume $\phi(E') \in X(E)$. Then by Pareto optimality of ϕ, we have

$$\max\{\phi_i(E), \phi_i(E')\} \leq \min\{p(R_i), p(R'_i)\} \quad \text{for all } i \in N. \tag{7.4}$$

Since $X(E) \subset X(E')$, it follows that

$$\min\{M, p(R_i)\} \leq p(R'_i) \quad \text{for all } i \in N. \tag{7.5}$$

Let $i \in N$ be such that $\phi_i(E) = p(R_i)$. Then it follows from (7.4) and (7.5) that $\phi_i(E') \leq p(R_i) = \phi_i(E) \leq p(R'_i)$. This implies that $\phi_i(E)\ R'_i\ \phi_i(E')$ for all $i \in N$ with $\phi_i(E) = p(R_i)$.

Since ϕ satisfies one-sided restricted monotonicity, it follows that $\phi_i(E')\ R'_i\ \phi_i(E)$ for all $i \in N$. Since $\phi(E) \in X(E')$, we must have $\phi_i(E')\ I'_i\ \phi_i(E)$ for all $i \in N$. Since both $\phi(E)$ and $\phi(E')$ are efficient in E', it follows that $\phi(E) = \phi(E')$.

Case 2: $\max\{\sum_{i\in N} p(R_i),\ \sum_{i\in N} p(R'_i)\} \leq M$.

In this case $X(E) \subset X(E')$ implies that $p(E') \leq p(E)$. Let $i \in N$ be such that $\phi_i(E) = p(R_i)$. Then, since $\phi(E') \in X(E)$, it follows from Pareto optimality of ϕ that $\phi_i(E') \geq p(R_i) = \phi_i(E) \geq p(R'_i)$. This implies that $\phi_i(E)\ R'_i\ \phi_i(E')$ for all $i \in N$ with $\phi_i(E) = p(R_i)$.

Since ϕ satisfies one-sided restricted monotonicity, it follows that $\phi_i(E')\ R'_i\ \phi_i(E)$ for all $i \in N$. Since both $\phi(E)$ and $\phi(E')$ are efficient in E', it follows that $\phi(E) = \phi(E')$.

$\square$

Lemma 7.3.5 will allow us to considerably simplify notation.

Lemma 7.3.5 Let ϕ be a Pareto optimal rule. If ϕ satisfies one-sided IIA, then ϕ satisfies peak only.

Proof. Let $E =< M, (R_i)_{i\in N} >$ and $E' =< M, (R'_i)_{i\in N} > \in \mathcal{E}$ be two economies with $p(E) = p(E')$. Then $X(E) = X(E')$ and since ϕ is Pareto optimal, we have $\phi(E') \in X(E) = X(E')$. Hence, by one-sided IIA, $\phi(E) = \phi(E')$. $\square$

Lemmas 7.3.4 and 7.3.5 imply the following corollary.

Corollary 7.3.6 Let ϕ be a Pareto optimal rule. If ϕ satisfies one-sided restricted monotonicity, it also satisfies peak only.

It will follow from Theorem 7.4.1 and from Example (III) in the next section that the converses of Lemmas 7.3.4, 7.3.5 and Corollary 7.3.6 are not true.

We conclude this section with the following property which imposes a restriction only when the solution satisfies peak only.

> *Conditional p-continuity:* A solution ϕ is conditional p-continuous if the following holds: If ϕ is peak only, then it is continuous with respect to the peaks.

Clearly, the uniform rule satisfies conditional p-continuity, which is weaker than the continuity with respect to preferences introduced in Sprumont (1991).

7.4 Two characterizations of the uniform rule

We are now ready to state the two main results of this chapter, which are characterizations of the uniform rule, based on the properties introduced in the previous section.

Theorem 7.4.1 The uniform rule is the unique rule which satisfies

(i) Pareto optimality

(ii) equal treatment

(iii) one-sided IIA

(iv) conditional p-continuity.

Proof. It is clear that the uniform rule satisfies (i)-(iv).
For each $M \in [0, \overline{M}]$, let $\mathcal{E}(M)$ be the class of economies in which M is the amount to be divided. Furthermore, let $\Delta(M) := \{x \in \mathbf{R}_+^N \mid x(N) = M\}$. For $p \in \mathbf{R}_+^N$ let $S(p) := \{x \in \Delta(M) \mid x \leq p \text{ or } x \geq p\}$.
Now let ϕ be a rule satisfying the foregoing properties. By Lemma 7.3.5 ϕ is peak only. Let $M \in [0, \overline{M}]$, and let $N \subset \mathbf{N}$ be a finite set of agents. Define the function $f : \mathbf{R}_+^N \to \Delta(M)$ by

$$f(p) = \phi(E) \text{ for some } E \in \mathcal{E}(M) \text{ with } p(E) = p.$$

Since ϕ is peak only, f is well-defined.
Since ϕ satisfies (i)-(iv), the reader can easily verify that f satisfies the following properties:

(A.1) $f(p) \in S(p)$ for all $p \in \mathbf{R}_+^N$.

(A.2) $f_i(p) = f_j(p)$ for all $p \in \mathbf{R}_+^N$ with $p_i = p_j$.

(A.3) For all $p, q \in \mathbf{R}_+^N$ such that either $\max\{p(N), q(N)\} < M$ or $\min\{p(N), q(N)\} \geq M$, we have, if $f(q) \in S(p) \subset S(q)$, then $f(p) = f(q)$.

(A.4) f is continuous in p.

To conclude the proof of Theorem 7.4.1 it suffices to show that for all $p \in \mathbf{R}_+^N$

$$f_i(p) = \begin{cases} \min\{p_i, \lambda\} & \text{if } p(N) \geq M \\ \max\{p_i, \lambda\} & \text{if } p(N) \leq M, \end{cases} \tag{7.6}$$

where λ is such that $f(p) \in \Delta(M)$.

Let $p \in \mathbf{R}_+^N$. Assume $p(N) < M$ (the case $p(N) > M$ is similar, and the case $p(N) = M$ is trivial). (A.1) implies $f(p) \geq p$.
Define the following set of agents:

$$K := \{i \in N \mid f_i(p) > p_i\}.$$

Since $p(N) < M$, $K \neq \emptyset$.

The proof of (7.6) follows from the following four lemmas.

Lemma A: Let $i \in K$ and let $0 \le q_i \le p_i$. Define $q \in \mathbf{R}_+^N$ by

$$q_j = \begin{cases} p_j & \text{if } j \in N \setminus \{i\} \\ q_i & \text{if } j = i. \end{cases}$$

Then $f(q) = f(p)$.

Proof. Let

$$\alpha := \inf\{z_i \in [q_i, p_i] \mid f(z_i, p_{-i}) = f(p)\}.$$

Here, p_{-i} denotes the vector $p_{N\setminus\{i\}}$.

By (A.4), it follows that $f(\alpha, p_{-i}) = f(p)$. We prove that $\alpha = q_i$. Suppose, on the contrary, that $\alpha > q_i$. Since $f_i(\alpha, p_{-i}) = f_i(p) > p_i$, it follows from (A.4) that there exists an $q_i < a < \alpha$ close enough to α, such that $f_i(a, p_{-i}) > p_i$. By (A.1), we have $f_j(a, p_{-i}) \ge p_j$ for all $j \in N \setminus \{i\}$. Hence, $f(a, p_{-i}) \in S(p)$. Clearly, $S(p) \subset S(a, p_{-i})$. Therefore, by (A.3) we have $f(a, p_{-i}) = f(p)$, contradicting the definition of α. We conclude that $\alpha = q_i$, and so it follows that $f(q) = f(q, p_{-i}) = f(p)$. □

Lemma B: for all $i, j \in K$ we have $f_i(p) = f_j(p)$.

Proof. Let $i, j \in K$, and let $0 \le v = \min\{p_i, p_j\}$. Define $q \in \mathbf{R}_+^N$ by

$$q_k = \begin{cases} p_k & \text{if } k \in N \setminus \{i, j\} \\ v & \text{if } k = i, j. \end{cases}$$

(A.2) yields that $f_i(q) = f_j(q)$. From Lemma A it now follows that $f_i(p) = f_j(p)$. □

Lemma C: For all $i, j \in N$ we have, if $p_i \le p_j$, then $f_i(p) \le f_j(p)$.

Proof. Suppose that there exist $i, j \in N$, with $p_i \le p_j$ and $f_i(p) > f_j(p)$. Define $q \in \mathbf{R}_+^N$ by

$$q_k = \begin{cases} p_k & \text{if } k \in N \setminus \{i\} \\ p_j & \text{if } k = i. \end{cases}$$

From (A.1) and the definition of q it follows that $q_k = p_k \le f_k(p)$ for $k \in N \setminus \{i\}$. Moreover, from the assumption it follows that $q_i = p_j \le f_j(p) < f_i(p)$. Hence, $f(p) \ge q$, and $\sum_{k \in N} q_k < \sum_{k \in N} f_k(p) = M$. Therefore, $f(p) \in S(q) \subset S(p)$. (A.3) now yields that $f(p) = f(q)$. Hence using (A.2), we obtain $f_i(p) = f_i(q) = f_j(q) = f_j(p)$, which contradicts the assumption $f_i(p) > f_j(p)$. □

According to Lemma B all agents in K obtain the same amount. Denote this amount by λ, i.e., $f_i(p) = \lambda$ for all $i \in K$.

Lemma D: $p_i \geq \lambda$ for all $i \in N \setminus K$.

Proof. Suppose that there exists an $i \in N \setminus K$, with $p_i < \lambda$. Take $j \in K$. By definition of K and λ we have $f_i(p) = p_i < \lambda = f_j(p)$. Hence by Lemma C, we have $p_i < p_j$. Define $q \in \mathbf{R}_+^N$ by

$$q_k = \begin{cases} p_k & \text{if } k \in N \setminus \{j\} \\ p_i & \text{if } k = j. \end{cases}$$

By Lemma A it follows that $f(q) = f(p)$. (A.2) yields $f_j(p) = f_j(q) = f_i(q) = f_i(p) = p_i < p_j$, which contradicts (A.1). □

Now we show that (7.6) holds.

From Lemma B and the definitions of K and λ it follows that

$$f_i(p) = \max\{p_i, \lambda\} \quad \text{for all } i \in K.$$

From Lemma D and the definition of K we obtain

$$f_i(p) = \max\{p_i, \lambda\} \quad \text{for all } i \in N \setminus K.$$

Since $f(p) \in \Delta(M)$, (7.6) holds. This completes the proof of Theorem 7.4.1. □

The following examples show that the properties (i)-(iv) in Theorem 7.4.1 are independent.

(I) The egalitarian rule satisfies equal treatment, (one-sided) IIA, and conditional p-continuity, but not Pareto optimality.

(II) Let ϕ^1 be defined as follows: For each $E =< M, (R_i)_{i \in N} >\in \mathcal{E}$

$$\phi^1(E) := \begin{cases} U(E) & \text{if } |N| \neq 2 \\ \text{argmax}\{x_i^{1/4} x_j^{3/4} \mid x \in X(E)\} & \text{if } N = \{i, j\},\ i < j. \end{cases}$$

ϕ^1 satisfies Pareto optimality, (one sided) IIA, and conditional p-continuity, but not equal treatment.

(III) Let ϕ^2 be defined as follows: For each $E =< M, (R_i)_{i\in N} >\in \mathcal{E}$ and $i \in N$

$$\phi_i^2(E) := \begin{cases} U_i(E) & \text{if } \sum_{j\in N} p(R_j) \geq M \\ p(R_i) + \frac{1}{|N|}(M - \sum_{j\in N} p(R_j)) & \text{otherwise.} \end{cases}$$

ϕ^2 satisfies Pareto optimality, equal treatment, and conditional p-continuity, but not (one-sided) IIA.

(IV) Let ϕ^3 be defined as follows: For each $E =< M, (R_i)_{i\in N} >\in \mathcal{E}$

$$\phi^3(E) := \begin{cases} U(E) & \text{if } |N| \neq 2 \\ \operatorname{argmin}\{x_i x_j \mid x \in X(E)\} & \text{if } N = \{i, j\} \text{ and } (\frac{M}{2}, \frac{M}{2}) \notin X(E) \\ (\frac{M}{2}, \frac{M}{2}) & \text{otherwise.} \end{cases}$$

ϕ^3 satisfies Pareto optimality, equal treatment, and (one-sided) IIA, but not conditional p-continuity.

Theorem 7.4.2 shows that if one-sided IIA is replaced by one-sided restricted monotonicity in Theorem 7.4.1, then we can drop conditional p-continuity.

Theorem 7.4.2 The uniform rule is the unique rule on $\mathcal{E}$ which satisfies

(i) Pareto optimality

(ii) equal treatment

(iii) one-sided restricted monotonicity.

Proof. It is clear that the uniform rule satisfies (i)-(iii). Now let ϕ be a rule satisfying (i)-(iii). By Corollary 7.3.6 ϕ is peak only. Let $M \in [0, \overline{M}]$, and let $N \subset \mathbf{N}$ be finite set of agents. Analogously to the proof of Theorem 7.4.1, define the function $f : \mathbf{R}_+^N \to \Delta(M)$ by

$$f(p) = \phi(E) \text{ for some } E \in \mathcal{E}(M) \text{ with } p(E) = p.$$

Since ϕ is peak only, f is well-defined.

The reader can easily verify that (i)-(iii) together with Lemma 7.3.4 imply that f satisfies, besides (A.1), (A.2), and (A.3) (see the proof of Theorem 7.4.1), the following property.

(A.5) For all $p, q \in \mathbf{R}_+^N$, with $S(p) \subset S(q)$, and such that either $\max\{p(N), q(N)\} < M$ or $\min\{p(N), q(N)\} \geq M$, we have, if $|f_i(p) - q_i| \leq |f_i(q) - q_i|$ for all $i \in N$ such that $f_i(p) = p_i$, then $|f_i(q) - q_i| \leq |f_i(p) - q_i|$ for all $i \in N$.

To conclude the proof of Theorem 7.4.2 it suffices to show that for all $p \in \mathbf{R}_+^N$

$$f_i(p) = \begin{cases} \min\{p_i, \lambda\} & \text{if } \sum_{i \in N} p_i \geq M \\ \max\{p_i, \lambda\} & \text{if } \sum_{i \in N} p_i \leq M, \end{cases}$$

where λ is such that $f(p) \in \Delta(M)$.

Let $p \in \mathbf{R}_+^N$. Assume $p(N) < M$ (the case $p(N) > M$ is similar, and the case $p(N) = M$ is trivial). (A.1) implies $f(p) \geq p$.
Define the following set of agents:

$$K := \{i \in N \mid f_i(p) > p_i\}.$$

Since $\sum_{i \in N} p_i < M$, $K \neq \emptyset$.
Analogously to the proof of Theorem 7.4.1 we now have the following lemma.
Lemma A': Let $i \in K$ and let $0 \leq q_i \leq p_i$. Define $q \in \mathbf{R}_+^N$ by

$$q_k = \begin{cases} p_k & \text{if } k \in N \setminus \{i\} \\ q_i & \text{if } k = i. \end{cases}$$

Then $f(q) = f(p)$.
Proof. From the definition of q it follows that $S(p) \subset S(q)$. Suppose $f(p) \neq f(q)$. Since $p, q \in \Delta(M)$, it is not true that $f(q) \leq f(p)$. Hence by (A.5), it follows that there exists a $j \in N$ such that $f_j(p) = p_j$ and $f_j(p) > f_j(q)$. Since $j \notin K$, it follows that $j \neq i$. Hence, $q_j = p_j = f_j(p) > f_j(q) \geq q_j$ which is a contradiction. □

The proof of Theorem 7.4.2 now follows from the remark that in the proofs of Lemmas B,C,D above only (A.1), (A.2) and (A.3) are used. So the proof of Theorem 7.4.2 can proceed in the same way as that of Theorem 7.4.1. □

The Examples (I) (II) and (III) above show that in Theorem 7.4.2 the characterizing properties are independent.

7.5 An application to a cost-sharing model

In this section we show that the results of Section 7.4 can also be applied in other models. As an example we discuss a classical cost-sharing model in which the costs of an indivisible public good (for example a bridge) have to be allocated among the agents. The only available data are the costs $c > 0$ of the public facility and the benefit $b_i \geq 0$ of each agent $i \in \{1, \ldots, n\}$. It is assumed that it is efficient to build the facility, so $\sum_{i=1}^n b_i \geq c$. The problem is how to share the costs.
Formally, the *cost-sharing problem* is described by a pair (c, b), where $c > 0$ and $b \in \mathbf{R}_+^n$ is such that $\sum_{i=1}^n b_i \geq c$. An *allocation* for (c, b) is a vector $x \in \mathbf{R}^n$ such that $\sum_{i=1}^n x_i = c$.
For this model many different allocation methods, which in this framework are called mechanisms, have been investigated (cf. Moulin (1988)). One of these mechanisms is the so-called *head tax*. The name head tax is introduced in Young (1987), who interprets the amount c as the total amount of tax needed to finance the public facility which is provided by the tax collector. For a problem (c, b) the head tax is defined by

$$h_i(c, b) := \min\{\lambda, b_i\} \qquad \text{for all } i = 1, \ldots, n$$

where $\lambda \geq 0$ is such that $\sum_{i=1}^n h_i(c, b) = c$.
Although the cost-sharing model is conceptually different from the previously discussed fair division problems with single-peaked preferences, there is a clear resemblance between both problems. If we associate to a cost-sharing problem (c, b) an economy $E = < M, (R_i)_{i=1}^n >$ with $M := c$ and $(R_i)_{i=1}^n$ are such that $p(E) = b$, then $U(E) = h(c, b)$.
As a consequence of this similarity Theorem 7.4.1 can be reformulated for the cost-sharing model. If we translate the Pareto optimality condition for economies, we obtain a property which is known as the core property (cf. Moulin (1988)).

> *Core property*: A mechanism f satisfies the core property if $0 \leq f(c, b) \leq b$ for all cost sharing problems (c, b).

The reason why Moulin (1988) called this property the core property is that, if to each cost-sharing problem a natural 'surplus sharing' TU-game is associated, then a payoff vector belongs to the core of this TU-game if and only if the corresponding cost allocation assigns to each agent a nonnegative cost which is smaller than his benefit. For a cost-sharing problem (c, b), we define the *core* of (c, b) as the set

$$C(c,b) := \{x \in \mathbf{R}^n \mid \sum_{i=1}^{n} x_i = c,\ 0 \leq x \leq b\}.$$

The independence of irrelevant alternatives property can now be formulated as follows.

> *IIA*: A mechanism f satisfies IIA if for all cost-sharing problems (c, b) and (c, b') with $C(c,b) \subset C(c,b')$, we have, if $f(c,b') \in C(c,b)$, then $f(c,b) = f(c,b')$.

Without proof we state the following analogue of Theorem 7.4.1.

Theorem 7.5.1 The head tax is the unique mechanism for cost-sharing problems which satisfies the core property, anonymity, IIA, and continuity with respect to the benefits.

In the context of bankruptcy problems (cf. Section 2.5) the uniform rule or head tax analogue is the so-called *constraint equal award rule*. Theorem 7.5.1 can easily be reformulated to obtain a characterization of this rule.

Part III

Effectivity functions

Chapter 8

Effectivity functions and game forms

A central topic in social choice theory is the analysis of collective decision rules called social choice correspondences or social choice functions. These decision rules reflect in some way or another the preferences of the individual agents over a certain set of alternatives.

The basis for social choice theory was laid more than two-hundred years ago by Borda (1781) and Condorcet (1785) who studied voting procedures for elections. The starting point for modern social choice theory is the contribution of Arrow (1951).

It has already been noted by Borda that in voting situations strategic aspects play an important role. By manipulating or misrepresenting preferences individuals or coalitions can influence society's choice. The famous Gibbard-Satterthwaite theorem (Gibbard (1973), Satterthwaite (1975)) shows that, when there are more than two alternatives, there are no non-dictatorial social choice functions which are non-manipulable (strategy-proof).

Given the fact that strategic aspects play an important role in social choice, it is interesting to study the power distribution in society, induced by the possibilities for individuals or coalitions to manipulate a given collective decision rule. A way to model this power distribution was introduced in Moulin and Peleg (1982) using the concept of an effectivity function. Formally an effectivity function associates to each coalition a collection of subsets of alternatives for which the coalition is *effective*. If a coalition is effective for a certain subset of alternatives this means that it can force the final outcome which has to be chosen to be an element of this set, or formulated

otherwise, it can *veto* all alternatives outside this set of alternatives.
The study of effectivity functions is the topic of Part III of this thesis. We will focus attention on the mathematical aspects of effectivity functions and do not go into a detailed description of the relation between effectivity functions and social choice theory. The reader is referred to Moulin and Peleg (1982), Moulin (1983), Peleg (1984a), and Abdou and Keiding (1992) for comprehensive studies on this subject.
This chapter presents an overview of some important results in the framework of effectivity functions. First, in Section 8.1 we present some basic notions and introduce effectivity functions associated with game forms. In Section 8.2 we formally define the concept of an effectivity function and discuss some basic properties of effectivity functions. Furthermore, we present some preliminary results which will be used in later chapters. In Section 8.3 the notion of the core of an effectivity function (at a certain preference profile) is defined, and the problem of non-emptiness of the core is addressed. Section 8.4 considers some special classes of effectivity functions which play an important role in the literature. Particularly, we discuss additive effectivity functions, effectivity functions corresponding to veto functions and effectivity functions corresponding to simple games. Finally, Section 8.5 describes relations between effectivity functions and game forms.

8.1 Social choice and effectivity functions

Let A be a finite set of alternatives and let N be the set $\{1, \ldots, n\}$ ($n \in \mathbb{N}$). N is called a *society*, members of N are called *agents* or *voters*, and non-empty subsets of N are called *coalitions*. Although we assume that A is finite, many of our results also hold in case A is infinite.
A central problem in social choice theory is how the society should make a choice from the given set of alternatives. The main idea is that society's choice should reflect the individual preferences of the agents. We assume that each agent $i \in N$ has preferences over the set of alternatives which can be described by a complete and transitive preference relation R_i (cf. Section 6.1). As in Part II of this monograph we use the notation $a\ R_i\ b$ if $(a,b) \in R_i$, and $a\ P_i\ b$ if $a\ R_i\ b$ and not $b\ R_i\ a$. Here $a\ R_i\ b$ is to be interpreted as 'alternative a is at least as good as alternative b according to agent i. By $\mathcal{R}$ we denote the set of all preference relations over A. Furthermore, we denote $R_S := (R_i)_{i \in S}$, $P_S := (P_i)_{i \in S}$ ($S \in \mathcal{P}_0(N)$). R_N is called a (preference) *profile*

on A. The set of all profiles on A is denoted by $\mathcal{R}_N$.

A *social choice function* is a surjective function $\phi : \mathcal{R}_N \to A$ (i.e., $\phi(\mathcal{R}_N) = A$). A *social choice correspondence* is a map ϕ which associates with each profile $R_N \in \mathcal{R}_N$ a non-empty subset $\phi(R_N)$ of A. Similar to surjectiveness of a social choice function we assume that a social choice correspondence is *non-imposed*, i.e., for each $a \in A$ there exists a profile $R_N \in \mathcal{R}_N$ such that $\phi(R_N) = \{a\}$. This assumption is not very restrictive, since alternatives that cannot be chosen even if all agents cooperate can be viewed as not really available to society.
A well-known example of a social choice correspondence is the *majority rule*, which selects the alternatives that are preferred by most agents. We do not give the formal definition of this social choice correspondence but we illustrate the majority rule by means of the following example.

Example 8.1.1 Let $N = \{1, 2, 3, 4, 5\}$ and $A = \{a, b, c, d\}$. Suppose the preferences of the agents on A are given by $R_N = (R_1, R_2, R_3, R_4, R_5)$, where

$R_1 = a\ c\ b\ d,$

$R_2 = c\ d\ a\ b,$

$R_3 = c\ b\ d\ a,$

$R_4 = d\ c\ a\ b,$

$R_5 = d\ a\ b\ c.$

Here there are no indifferences and the preferences of the players are denoted in decreasing order, i.e. player 1 likes a the most, then c, then b, and then d, and so on. We see that there is only one agent who has a as his best alternative (agent 1), there is no agent who has b as his most preferred alternative, there are two agents who prefer alternative c above all other alternatives (agents 2 and 3), and there are also two agents who prefer d (agents 4 and 5). The majority rule chooses the alternatives which have most first places. Since c and d are most preferred by two agents, while a and b are most preferred by less agents, the majority rule yields $\{c, d\}$. This example illustrates that in general the majority rule does not select a single alternative as an outcome. So the majority rule is a social choice correspondence and not a social choice function.

It is interesting to notice that a social choice correspondence (or function) induces a power distribution in society. Given a fixed social choice correspondence, agents or coalitions have the possibility to influence the outcome prescribed by the social choice correspondence by manipulating, i.e., reporting other than their true preferences. We illustrate this phenomenon in the following example.

Example 8.1.2 Consider the situation described in Example 8.1.1. We have seen that the majority rule does not necessarily select a single alternative as an outcome. In order to overcome this difficulty we adopt a tie-breaking rule. If the majority rule does not select a single alternative, then alternative d is chosen as the outcome. In Example 8.1.1 the majority rule yields $\{c, d\}$, so application of the tie-breaking rule implies that alternative d will be selected. Agent 1, for example, would have preferred alternative c, since d is his worst alternative. He can achieve this goal by misrepresenting his preferences. If he reports $R_1' = c\ a\ b\ d$ instead of his (true) preferences R_1, then the majority rule selects the single alternative $\{c\}$ as an outcome, so the tie-breaking rule need not be applied.

Example 8.1.2 shows that by reporting preferences strategically, agents have the power to affect society's choice. This undesirable property of manipulability is not a specific drawback of the (adopted) majority rule. The classical theorem of Gibbard-Satterthwaite illustrates that essentially there are no non-dictatorial social choice functions which are non-manipulable.
Since we are interested in the power of coalitions rather than the power that individual agents may have, it will be clear that if groups of agents coordinate their actions, there are even more possibilities for manipulation.

A natural framework to further investigate the strategic aspects of the problem of collective choice is provided by non-cooperative game theory. First, we introduce some additional notation.
Let $S \in \mathcal{P}_0(N)$ and for all $i \in S$, let X_i be a non-empty set. We denote the Cartesian product $\prod_{i \in S} X_i$ by X_S. If $\sigma_i \in X_i$ for all $i \in S$, then we write σ_S instead of $(\sigma_i)_{i \in S}$. A *game form* (Gibbard (1973)) is an $(n+2)$-tuple $G = (X_1, \ldots, X_n, A, \pi)$, where X_i is a non-empty set of *strategies* for each $i \in N$, A is a finite set of alternatives, and $\pi : X_N \to A$ is a surjective *outcome function* (i.e., $\pi(X_N) = A$).
The interpretation of G is as follows: Given the choice $\sigma_i \in X_i$ of each player $i \in N$, the outcome function π determines an alternative $\pi(\sigma_N) \in A$. Note the difference between a game form and a *game in strategic form* as studied already by von Neumann and Morgenstern (1944) and Nash (1951). Each game form gives rise to a game in normal form by adding to the game form the preferences of the agents. (Usually in

the definition of a game in strategic form it is assumed that the preferences of the agents are represented by utility functions.)

A social choice function $\phi : \mathcal{R}_N \to A$ gives rise to a game form in the following way: The strategy set of each agent is $\mathcal{R}$, and the outcome function is ϕ. This means that we have embedded the problem of collective power distribution in society induced by a social choice function in the more general framework of collective power distribution induced by a game form.

In the literature several ways to measure collective power induced by a game form have been proposed. Hurwicz and Schmeidler (1978) introduced a simple game which specifies all winning coalitions, i.e., coalitions which have the power to enforce every single alternative as an outcome, independent of what the other agents do. Ishikawa and Nakamura (1980) considered a kind of game in characteristic function form, which can be regarded as a generalization of the simple game of Hurwicz and Schmeidler. However, these two approaches only give a partial description of the power distribution.

A more general way to describe coalitional power induced by a game form is introduced in Moulin and Peleg (1982) using the notion of an effectivity function. An effectivity function specifies for each coalition the collection of subsets of alternatives for which that coalition is *effective*. If a coalition is effective for a certain subset of alternatives this means that it can guarantee the final outcome to be within this subset.

Several effectivity functions can be associated with a given game form. We restrict attention to the two possibilities that received most attention in the literature.

Let $G = (X_1 \ldots, X_n, A, \pi)$ be a game form, let $S \in \mathcal{P}_0(N)$ be a coalition and let $B \in \mathcal{P}_0(A)$ be a non-empty subset of alternatives. S is called *α-effective* for B if there exists $\sigma_S \in X_S$ such that $\pi(\sigma_S, \tau_{N\setminus S}) \in B$ for all $\tau_{N\setminus S} \in X_{N\setminus S}$. S is called *β-effective* for B if for all $\tau_{N\setminus S} \in X_{N\setminus S}$ there exists $\sigma_S \in X_S$ such that $\pi(\sigma_S, \tau_{N\setminus S}) \in B$. By $E^G_\alpha(S)$ ($E^G_\beta(S)$) we denote the set of all subsets of alternatives for which S is α-effective (β-effective). The functions $E^G_\alpha, E^G_\beta : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ are called the *α-effectivity function and the β-effectivity function associated with the game form G.* Summarizing we have that for $S \in \mathcal{P}_0(N)$

$$E^G_\alpha(S) := \{B \in \mathcal{P}_0(A) \mid \exists \sigma_S \in X_S \; \forall \tau_{N\setminus S} \in X_{N\setminus S} : \; \pi(\sigma_S, \tau_{N\setminus S}) \in B\} \text{ and}$$

$$E^G_\beta(S) := \{B \in \mathcal{P}_0(A) \mid \forall \tau_{N\setminus S} \in X_{N\setminus S} \; \exists \sigma_S \in X_S : \; \pi(\sigma_S, \tau_{N\setminus S}) \in B\}.$$

From these definitions it is immediately clear that

(i) $E_\alpha^G(N) = E_\beta^G(N) = \mathcal{P}_0(A)$, and

(ii) $A \in E_\alpha^G(S)$, $A \in E_\beta^G(S)$ for all $S \in \mathcal{P}_0(N)$,

where (i) follows from the fact that the outcome function π is surjective.
The α- and β-effectivity function associated with a game form are introduced in Moulin and Peleg (1982), but the idea of α- and β-effectiveness of coalitions goes back to the fifties (cf. Aumann (1961)). Note that E_α^G specifies all possibilities of pessimistic cooperative behavior of the agents, while E_β^G corresponds to a more optimistic view. It is clear that $E_\alpha^G(S) \subset E_\beta^G(S)$ for all $S \in \mathcal{P}_0(N)$ and all game forms G. The converse inclusion however need not be satisfied. A game form G is called *tight* if $E_\alpha^G = E_\beta^G$.

We conclude this section with an example which can be found in Moulin (1983).

Example 8.1.3 A way to determine a collective choice is the method of *voting by veto*. In this decision mechanism every agent $i \in N$ is equipped with a *veto power* $m_i \in \mathbb{N} \cup \{0\}$, and every alternative $a \in A$ has a *veto resistance* $r_a \in \mathbb{N}$. These numbers are such that the total veto resistance is one more than the total veto power, i.e.,

$$\sum_{i \in N} m_i = \sum_{a \in A} r_a - 1.$$

Let $A(r)$ be the set made up by r_a replicas of alternative a for all $a \in A$. Formally,

$$A(r) := \{(a, k) \mid a \in A,\ k \in \mathbb{N},\ k \leq r_a\}.$$

Further, let $\Sigma(m)$ be the set of all orderings in which every agent $i \in N$ appears exactly m_i times. So,

$$\Sigma(m) := \{\sigma : \{1, \ldots, \sum_{i \in N} m_i\} \to N \mid |\sigma^{-1}(i)| = m_i \text{ for all } i \in N\}.$$

Given an ordering $\sigma \in \Sigma(m)$, we define a game form (in extensive form) G_σ as follows.
In step 1 agent $\sigma(1)$ vetoes one element of $A(r)$, say (a_1, k_1).
In step 2 agent $\sigma(2)$ vetoes one element of $A(r) \setminus \{(a_1, k_1)\}$, say (a_2, k_2).
$\vdots$
In step t agent $\sigma(t)$ vetoes one element of $A(r) \setminus \{(a_1, k_1), \ldots, (a_{t-1}, k_{t-1})\}$, say (a_t, k_t).
$\vdots$

After step $\sum_{i \in N} m_i$, there is exactly one element of $A(r)$ which is not vetoed, say (a, k). The alternative a is the selected outcome.
Note that the above procedure corresponds to an extensive game form. However, in the standard way a strategic game form can be derived. Since every alternative can occur as the final outcome, it follows that the outcome function is surjective.
Moulin (1983) shows that this game form is tight and that its associated effectivity function $E_{m,r} := E_{\alpha}^{G_\sigma} = E_{\beta}^{G_\sigma}$ is (independently of the order σ) given as follows. For $S \in \mathcal{P}_0(N)$ and $B \in \mathcal{P}_0(A)$,

$$B \in E_{m,r}(S) \text{ if and only if } \sum_{i \in S} m_i \geq \sum_{a \in A \setminus B} r_a.$$

Or equivalently,

$$B \in E_{m,r}(S) \text{ if and only if } \sum_{i \in S} m_i + \sum_{a \in B} r_a \geq \sum_{a \in A} r_a.$$

$E_{m,r}$ is called the *effectivity function associated with the voting by veto method* where $m := (m_i)_{i \in N}$ and $r := (r_a)_{a \in A}$ are the veto power and veto resistance, respectively.

8.2 The structure of effectivity functions

In the previous section we introduced effectivity functions corresponding to a game form as a way to measure the collective power distribution in a society induced by the game form. In this section we study effectivity functions on their own and present some preliminary results which will be used later on.

Based on the definition of effectivity functions associated with game forms in Section 8.1 we define the general notion of an effectivity function as follows.
An *effectivity function* is a map $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ such that

(i) $E(N) = \mathcal{P}_0(A)$

(ii) $A \in E(S)$ for all $S \in \mathcal{P}_0(N)$.

The interpretation of E is as follows: If $B \in E(S)$, then S can force the final decision within the subset B of alternatives. By definition the society N can force the outcome to belong to every (non-empty) subset of alternatives.

There are alternative ways to represent an effectivity function. We mention two of them.

An effectivity function E can be represented by means of a $\{0,1\}$-matrix I^E of size $2^n - 1$ by $2^{|A|} - 1$, where for $S \in \mathcal{P}_0(N)$ and $B \in \mathcal{P}_0(A)$,

$$I^E(S, B) = 1 \text{ if and only if } B \in E(S).$$

Alternatively, an effectivity function E can be seen as a collection $\{v_B\}_{B \in \mathcal{P}_0(A)}$ of simple TU-games, where for each $B \in \mathcal{P}_0(A)$ the game $v_B : 2^N \to \{0,1\}$ is defined by

$$v_B(S) = 1 \text{ if and only if } B \in E(S).$$

We will now consider several properties that effectivity functions might satisfy. We will use these properties later on, but it should be mentioned that this list of properties is certainly not exhaustive. For more properties of effectivity functions we refer the reader to Abdou and Keiding (1992).

Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be an effectivity function.

(i) E is *A-monotonic* if for all $S \in \mathcal{P}_0(N)$ and all $B, B' \in \mathcal{P}_0(A)$ with $B \subset B'$ and $B \in E(S)$, we have $B' \in E(S)$.

(ii) E is *N-monotonic* if for all $S, S' \in \mathcal{P}_0(N)$ with $S \subset S'$, and all $B \in \mathcal{P}_0(A)$, with $B \in E(S)$, we have $B \in E(S')$.

(iii) E is *maximal* if for all $S \in \mathcal{P}_0(N)$ and all $B \in \mathcal{P}_0(A)$ such that $B \notin E(S)$, we have $A \setminus B \in E(N \setminus S)$.

(iv) E is *neutral* if for all $S \in \mathcal{P}_0(N)$, all $B \in E(S)$ and all $B' \in \mathcal{P}_0(A)$ with $|B'| = |B|$, we have $B' \in E(S)$.

(v) E is *superadditive* if for all $S_1, S_2 \in \mathcal{P}_0(N)$, with $S_1 \cap S_2 = \emptyset$, and all $B_1 \in E(S_1)$, $B_2 \in E(S_2)$, we have $B_1 \cap B_2 \in E(S_1 \cup S_2)$.

(vi) E is *upper cycle free* if for all $S_1, \ldots, S_k \in \mathcal{P}_0(N)$ with $S_r \cap S_t = \emptyset$ for all $r, t \in \{1, \ldots, k\}$, $r \neq t$ and all $B_1, \ldots, B_k \in \mathcal{P}_0(A)$ with $B_r \in E(S_r)$ for all $r \in \{1, \ldots, k\}$, we have $\bigcap_{r=1}^{k} B_r \neq \emptyset$.

(vii) E is *convex* if for all $S_1, S_2 \in \mathcal{P}_0(N)$, and all $B_1 \in E(S_1)$, $B_2 \in E(S_2)$, we have $B_1 \cap B_2 \in E(S_1 \cup S_2)$ or $B_1 \cup B_2 \in E(S_1 \cap S_2)$.

It is easy to check that if E is convex, then E is also superadditive, and if E is superadditive, then E is upper cycle free and N-monotonic.
For every game form G the effectivity functions E_α^G and E_β^G are both A- and N-monotonic. Furthermore, E_α^G is superadditive and E_β^G is maximal. So if a game form G is tight, then E_α^G is maximal. That the converse also holds is shown in the following lemma which can be found in Peleg (1984a).

Lemma 8.2.1 Let G be a game form. Then G is tight if and only if E_α^G is maximal.

Since the game form G_σ introduced in Example 8.1.3 is tight it follows from the previous lemma that the effectivity function $E_{m,r}$ is maximal. In Section 8.4 we will see that $E_{m,r}$ is even convex.

The superadditive cover $\overline{E}$ of an effectivity function E is introduced in Peleg (1984a). In the remaining part of this section we establish some relations between E and $\overline{E}$ which will be used in Chapter 10.

Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be an effectivity function. The *superadditive cover* $\overline{E}$ *of* E is a map that assigns to each $S \in \mathcal{P}_0(N)$ a subset of 2^A such that it satisfies the following property: $B \in 2^A$ is an element of $\overline{E}(S)$ if and only if there exist a partition $\{S_1, \ldots, S_k\}$ of S and sets $B_r \in E(S_r)$ for all $r \in \{1, \ldots, k\}$ such that $B = \bigcap_{r=1}^k B_r$. So, $\overline{E}$ is a function from $\mathcal{P}_0(N)$ to 2^{2^A}. It is easy to see that $E \subset \overline{E}$ for all effectivity functions E, i.e., $E(S) \subset \overline{E}(S)$ for all $S \in \mathcal{P}_0(N)$. It is clear that $\overline{E}$ is an effectivity function ($\emptyset \notin \overline{E}(S)$ for all $S \in \mathcal{P}_0(N)$) if and only if E is upper cycle free. The name 'superadditive cover' is explained in the next two lemmas.

Lemma 8.2.2 Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be an effectivity function. The following three properties are equivalent.

(i) E is upper cycle free.

(ii) $\overline{E}$ is a superadditive effectivity function.

(iii) There exists a superadditive effectivity function E' such that $E \subset E'$.

Proof. (i)$\Rightarrow$(ii) Let E be upper cycle free. Then by definition $\emptyset \notin \overline{E}(S)$ for all $S \in \mathcal{P}_0(N)$. Hence $\overline{E}$ is an effectivity function. Let S, $T \in \mathcal{P}_0(N)$ with $S \cap T = \emptyset$ and let $B \in \overline{E}(S)$ and $D \in \overline{E}(T)$. Then there are partitions $\{S_1, \ldots, S_k\}$ of S and $\{T_1, \ldots, T_l\}$ of T and there are $B_1, \ldots, B_k$ and $D_1, \ldots, D_l$ with $B_r \in E(S_r)$ for all

$r \in \{1,\ldots,k\}$ and $D_s \in E(T_s)$ for all $s \in \{1,\ldots,l\}$ such that $B = \bigcap_{r=1}^k B_r$ and $D = \bigcap_{s=1}^l D_s$. Then $\{S_1,\ldots,S_k,T_1,\ldots,T_l\}$ is a partition of $S \cup T$ and therefore by definition of $\overline{E}$ we obtain $B \cap D \in \overline{E}(S \cup T)$. So $\overline{E}$ is superadditive.

(ii)$\Rightarrow$(iii) Trivial because $E \subset \overline{E}$.

(iii)$\Rightarrow$(i) Let E' be a superadditive effectivity function such that $E \subset E'$. Since E' is superadditive, E' is upper cycle free. But then E, too, is upper cycle free because $E \subset E'$. □

The following lemma shows that for an upper cycle free effectivity function E, $\overline{E}$ is the smallest superadditive effectivity function containing E.

Lemma 8.2.3 Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be an effectivity function. Then for each superadditive effectivity function E' with $E \subset E'$ we have $E \subset \overline{E} \subset E'$.

Proof. Let E' be a superadditive effectivity function such that $E \subset E'$. Since $E \subset E'$ and E' is superadditive, it follows from Lemma 8.2.2 that E is upper cycle free and that $\overline{E}$ is superadditive. Let $S \in \mathcal{P}_0(N)$ and let $B \in \overline{E}(S)$. Then there is a partition $\{S_1,\ldots,S_k\}$ of S and there are $B_1,\ldots,B_k$ with $B_r \in E(S_r)$ for all $r \in \{1,\ldots,k\}$ such that $B = \bigcap_{r=1}^k B_r$. Then by superadditivity of E' and $E \subset E'$ it follows that $B \in E'(S)$. □

8.3 Stability of effectivity functions

Given an effectivity function that describes coalitional power in society and a profile reflecting the individual preferences of all agents, the problem of interest is how to find an alternative, or a set of alternatives, which every agent can agree upon. Since we study situations in which agents behave cooperatively, a rather natural solution concept is the core of an effectivity function (Moulin and Peleg (1982)). The core describes whether the situation is stable with respect to coalitional deviations.

Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be an effectivity function and $R_N \in \mathcal{R}_N$ a profile. An alternative $a \in A$ is *dominated* by a subset $B \in \mathcal{P}_0(A)$ of alternatives via a coalition $S \in \mathcal{P}_0(N)$ if $B \in E(S)$ and $b\ P_S\ a$ for all $b \in B$. The *core of E at R_N*, $Core(E, R_N)$, consists of all alternatives $a \in A$ which are not dominated by any subset of alternatives via any coalition. An effectivity function E is called *stable* if $Core(E, R_N) \neq \emptyset$ for all profiles $R_N \in \mathcal{R}_N$.

Stability of effectivity functions is a desirable requirement, since it excludes situations where every alternative that can be chosen is unstable with respect to coalitional opposition. Stability of effectivity functions has been studied by several authors. In this section we summarize some of the results.
The first general results on stability of effectivity are due to Peleg (1982), who showed that convex effectivity functions are stable.

Theorem 8.3.1 Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be a effectivity function.

(i) If E is convex, then E is stable.

(ii) If E is stable and maximal, then E is convex.

We will not give a formal proof of this theorem, but we illustrate the idea of Peleg's proof of (i). To prove (i), Peleg associated with each effectivity function and a preference profile an NTU-game (cf. Section 2.4). He proved that if E is convex, then the associated NTU-game is ordinal convex and hence has a non-empty core. The proof is concluded by showing that a core element of the NTU-game 'corresponds' to a core element of the effectivity function. Elementary proofs of (i) not making use of cooperative game theory have been provided by Ishiishi (1985) and Demange (1987).

Theorem 8.3.1 yields a characterization of the class of stable effectivity functions which are maximal. A characterization of stable effectivity functions is due to Keiding (1985) who introduced the notion of *acyclicity*. As it is very technical, we will not discuss this result and refer the interested reader to Keiding (1985).

8.4 Special classes of effectivity functions

In this section we discuss three subclasses of effectivity functions, all introduced in Moulin and Peleg (1982), which play an important role in the literature. Successively, we discuss effectivity functions corresponding to simple games, additive effectivity functions, and effectivity functions corresponding to veto functions.

Simple games

Let N be a society and A a finite set of alternatives. A *simple game* on N is a pair (N, v) (often denoted simply by v), where $v : 2^N \to \{0, 1\}$ is a function with $v(\emptyset) = 0$

and $v(N) = 1$. A simple game v is *monotonic* if for all $S, T \in \mathcal{P}_0(N)$ with $S \subset T$ and $v(S) = 1$ it holds that $v(T) = 1$. Let S be a coalition. If $v(S) = 1$, then S is a *winning* coalition, and if $v(S) = 0$, then S is called *losing*.

A way of associating an effectivity function $E^v : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ to a monotonic simple game (N, v) is the following. For $S \in \mathcal{P}_0(N)$

$$E^v(S) := \begin{cases} \mathcal{P}_0(A) & \text{if } S \text{ is winning} \\ \{A\} & \text{if } S \text{ is losing.} \end{cases}$$

Winning coalitions have the power to enforce every subset of alternatives, whereas a losing coalition has no power at all. In Peleg (1984a) this effectivity function is called the *standard effectivity function associated with* v. It is clear that E^v is A-monotonic and N-monotonic. Furthermore, if v is *proper*, i.e., $v(S) = 1$ implies $v(N \setminus S) = 0$, then E^v is superadditive, and if v is *strong*, i.e., $v(S) = 0$ implies $v(N \setminus S) = 1$, then E^v is maximal. Finally, if v is balanced (cf. Section 2.1), then E^v is stable. A complete characterization of stable effectivity functions associated with a monotonic simple games is provided by Nakamura (1979).

Additive effectivity functions

Let $\lambda \in \mathbf{R}^N$ and $\mu \in \mathbf{R}^A$ be two positive probability measures on N and A, respectively. So, $\lambda_i > 0$ for all $i \in N$ and $\sum_{i \in N} \lambda_i = 1$, and $\mu_a > 0$ for all $a \in A$ and $\sum_{a \in A} \mu_a = 1$. The vectors λ and μ give rise to an effectivity function $E_{\lambda,\mu}$ in the following way. For $S \in \mathcal{P}_0(N)$ and $B \in \mathcal{P}_0(A)$

$$B \in E_{\lambda,\mu}(S) \text{ if and only if } \sum_{i \in S} \lambda_i > \sum_{a \in A \setminus B} \mu_a.$$

Analogous to Example 8.1.3 the interpretation is that S is effective for B if the total veto power of S exceeds the total veto resistance of $A \setminus B$. Using the fact that $\sum_{a \in A} \mu_a = 1$ we see that

$$B \in E_{\lambda,\mu}(S) \text{ if and only if } \sum_{i \in S} \lambda_i + \sum_{a \in B} \mu_a > 1.$$

It is left to the reader to check that $E_{\lambda,\mu}$ is indeed an effectivity function. An effectivity function $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ is called *additive* if there exist positive probability measures $\lambda \in \mathbf{R}^N$ and $\mu \in \mathbf{R}^A$ such that $E = E_{\lambda,\mu}$. It is clear that these probability measures need not be uniquely determined.

Additive effectivity functions play a prominent role in the literature on effectivity functions. One of the reasons is that additive effectivity functions are stable. This is a consequence of the following theorem by Peleg (1982).

Theorem 8.4.1 Additive effectivity functions are convex.

An important application of additive effectivity functions is the class of effectivity functions corresponding to a voting by veto method (see Example 8.1.3). Moulin (1983) showed that each effectivity function $E_{m,r}$, associated with the voting by veto method where m and r are the veto power and veto resistance, is additive. The converse however, does not hold. Not each additive effectivity function can be considered as an effectivity function $E_{m,r}$ associated with a voting by veto method for certain m and r. This is due to the fact that additive effectivity functions need not be maximal (it can be shown that an additive effectivity function $E_{\lambda,\mu}$ is maximal if and only if $\lambda(S) + \mu(B) \neq 1$ for all $S \in \mathcal{P}_0(N)$ and all $B \in \mathcal{P}_0(A)$). If maximality is added then the converse implication does hold as is shown in Moulin (1983).
Storcken (1994) characterizes the class of additive effectivity functions by associating a simple game with each effectivity function and using Elgot's (1961) characterization of the class of all weighted simple games. For details on this result the reader is referred to Storcken (1994).

Veto functions

A *veto function* (cf. Moulin (1982)) is a function $\nu : 2^N \to \{0, 1, \ldots, |A| - 1\}$ with $\nu(\emptyset) := 0$ and $\nu(N) := |A| - 1$.
Given a veto function ν, the *effectivity function* E^ν *corresponding to* ν is defined by

$$E^\nu(S) := \{B \in \mathcal{P}_0(A) \mid \nu(S) \geq |A \setminus B|\}$$

for all $S \in \mathcal{P}_0(N)$.
We leave it to the reader to verify that E^ν is an effectivity function. Again, the interpretation is that a coalition S is effective for a subset of alternatives if S can veto all alternatives outside B, where the veto power of coalitions is described by the veto function ν (which is a TU-game). Since in this case the veto power of coalitions need not be additive, it is clear that effectivity functions corresponding to veto functions need not be additive effectivity functions.
Several properties of E^ν can be formulated in terms of the veto function ν (cf. Abdou and Keiding (1992)). For example, E^ν is superadditive if and only if ν is *superadditive*, i.e., $\nu(S_1 \cup S_2) \geq \nu(S_1) + \nu(S_2)$ for all $S_1, S_2 \in \mathcal{P}_0(N)$ with $S_1 \cap S_2 = \emptyset$. Further, E^ν is maximal if and only if $\nu(S) + \nu(N \setminus S) \geq |A| - 1$ for all $S \in \mathcal{P}_0(N)$. The following proposition shows that, analogous to effectivity functions associated with monotonic simple games, balancedness of ν is a sufficient condition for stability of E^ν.

Proposition 8.4.2 Let E^ν be the effectivity function corresponding to the veto function $\nu : \mathcal{P}_0(N) \to \{0, 1, \ldots, |A| - 1\}$. If ν is balanced, then E^ν is stable.

Proof. Suppose that $x \in C(\nu)$, i.e., $x(N) = |A| - 1$ and $x(S) \geq \nu(S)$ for all $S \in \mathcal{P}_0(N)$. Define positive probability measures $\lambda \in \mathbf{R}^N$ and $\mu \in \mathbf{R}^A$ by

$$\lambda_i := \frac{1}{|A|}(x_i + \frac{1}{n}) \text{ for all } i \in N, \tag{8.1}$$

$$\mu_a := \frac{1}{|A|} \text{ for all } a \in A. \tag{8.2}$$

Let $E_{\lambda,\mu}$ be the additive effectivity function generated by λ and μ.
Claim: $E^\nu(S) \subset E_{\lambda,\mu}(S)$ for all $S \in \mathcal{P}_0(N)$.
Proof of the claim: Let $S \in \mathcal{P}_0(N)$ and $B \in E^\nu(S)$. Then $\nu(S) \geq |A \setminus B|$, and so

$$\nu(S) + |B| \geq |A| > |A| - \frac{|S|}{n}.$$

Since $x(S) \geq \nu(S)$, it follows that

$$(x(S) + \frac{|S|}{n}) + |B| > |A|.$$

Using (8.1) and (8.2) we obtain

$$\lambda(S) + \mu(B) > 1.$$

Hence $B \in E_{\lambda,\mu}(S)$, which proves the claim.
Since $E_{\lambda,\mu}$ is stable (by Theorem 8.4.1 and Theorem 8.3.1), it directly follows that E^ν is also stable. □

Contrary to the class of additive effectivity functions, it is rather easy to characterize the class of effectivity functions corresponding to veto functions. The effectivity function E^ν corresponding to veto function ν is neutral and A-monotonic. Conversely, every neutral and A-monotonic effectivity function E generates a veto function ν^E defined by

$$\nu^E(S) := \max\{|A \setminus B| \mid B \in E(S)\}$$

for all $S \in \mathcal{P}_0(N)$, such that $E^{\nu^E} = E$.

Since effectivity functions associated with monotonic simple games are both neutral and A-monotonic, it follows that this class is a subclass of the effectivity functions corresponding to veto functions. Additive effectivity functions however, need not be

neutral, so this class is not a subclass of the class of effectivity functions corresponding to veto functions.
In Chapter 9 we introduce another class of effectivity functions, called decomposable effectivity functions, which incorporates all three classes of effectivity functions that we discussed in this section.

8.5 Game forms and effectivity functions

In Section 8.1 we have associated the α- and β-effectivity functions with a game form. It is interesting to investigate when a given effectivity function can be obtained as the α- or β-effectivity function associated with a game form. In Section 8.2 we have seen that for a game form G the effectivity function E_α^G is superadditive and A-monotonic. Moulin (1983) also proved the following converse result.

Theorem 8.5.1 Let E be an effectivity function. Then there exists a game form G such that $E_\alpha^G = E$ if and only if E is superadditive and A-monotonic.

In Section 10.4 we provide an alternative proof of this theorem.
Another result by Moulin (1983) is that every maximal and superadditive effectivity function can be obtained by a tight game form (note that A-monotonicity is not required since this is implied by superadditivity and maximality).

Theorem 8.5.2 Let E be an effectivity function. Then there exists a tight game form G such that $E_\alpha^G = E_\beta^G = E$ if and only if E is superadditive and maximal.

In Chapter 10 we associate with an effectivity function a game correspondence, i.e., a game form with an outcome correspondence instead of an outcome function. Among others we characterize the class of effectivity functions associated with tight game correspondences. It turns out that these effectivity functions need not be maximal.

In the last part of this section we examine relations between solution concepts of effectivity functions and game forms (at a certain preference profile). As noticed in Section 8.1 a game form and a profile give rise to a game in strategic form. For these kind of games many solution concepts have been investigated in the literature. The most prominent one is the concept of Nash equilibrium (Nash (1951)). A Nash equilibrium is robust against individual deviations of the agents. As our framework is of a cooperative nature, we are interested in solution concepts that are robust against coordinated deviations of coalitions. For our purposes an appropriate modification of the Nash equilibrium concept is the notion of strong Nash equilibrium (cf. Aumann (1959)).

Let $G = (X_1, \ldots, X_n, A, \pi)$ be a game form and R_N a profile on A. A strategy vector $\sigma_N \in X_N$ is called a *strong Nash equilibrium* of G at R_N if there do not exist $S \in \mathcal{P}_0(N)$ and $\tau_S \in X_S$ such that $\pi(\tau_S, \sigma_{N \setminus S})\ P_S\ \pi(\sigma_N)$. By $SNE(G, R_N)$ we denote the set of all strong Nash equilibria of G at R_N. G is called *strongly consistent* (cf. Moulin and Peleg (1982)) if $SNE(G, R_N) \neq \emptyset$ for all $R_N \in \mathcal{R}_N$.
Moulin and Peleg (1982) proved that if G is a game form and R_N is a profile, then each strong Nash equilibrium of G at R_N yields an outcome which belongs to the core of the α-effectivity function associated with G. Moreover, they showed that a strongly consistent game form is tight. So if G is a strongly consistent game form, then E_α^G $(= E_\beta^G)$ is stable and maximal. The following converse implication is also due to Moulin and Peleg (1982).

Theorem 8.5.3 Let E be a stable and maximal effectivity function and let R_N be a profile. Then there exists a strongly consistent game form G such that

(i) $E = E_\alpha^G = E_\beta^G$ and

(ii) $Core(E, R_N) = \pi(SNE(G, R_N))$.

In Chapter 10 we show that if we consider game correspondences and associated effectivity functions, then we can obtain a counterpart of Theorem 8.5.3 for a larger class of effectivity functions.

Chapter 9

Decomposable effectivity functions

In Section 8.4 we discussed three special classes of effectivity functions which play a prominent role in the literature on effectivity functions. We have seen that effectivity functions associated with monotonic simple games form a subclass of the class of effectivity functions corresponding to veto functions and that there is no inclusion relation between the class of additive effectivity functions and the class of effectivity functions corresponding to veto functions. In this chapter, which is based on Otten, Borm, Storcken and Tijs (1995), we introduce another class of effectivity functions, called decomposable effectivity functions, which comprises the classes mentioned above.

In Section 9.1 we introduce decomposable effectivity functions as a natural extension of additive effectivity functions where the veto power of coalitions and the veto resistance of subsets of alternatives can be described by TU-games being not necessarily additive.

In Section 9.2 we examine relations between the properties of decomposable effectivity functions and the properties of TU-games that generate these effectivity functions. Among others, it is shown that a decomposable effectivity function is monotonic if and only if it can be generated by monotonic TU-games and, that a decomposable effectivity function is stable whenever it can be generated by balanced TU-games.

Sections 9.3 and 9.4 provide two characterizations of the class of decomposable effectivity functions. First, it is shown that an effectivity function is decomposable if and only if it satisfies the revealed power property. This property can be seen as a modification of the more familiar WARP condition (=weak axiom of revealed preference) in social choice theory. Next, we show that an effectivity function is decomposable if and only if it is possible to represent the effectivity function by a $\{0,1\}$-matrix in echelon form.

9.1 Decomposable effectivity functions

Additive effectivity functions correspond to situations where the veto power is distributed in an additive way among the coalitions, i.e., the total veto power of the union of two disjoint coalitions equals the sum of the veto powers of the separate coalitions. Also the veto resistance of subsets of alternatives is distributed in an additive way. The reason that effectivity functions corresponding to veto functions need not be additive is that the veto power of coalitions is described by a veto function which need not be additive.

In this section we introduce a generalization of additive effectivity functions, which also incorporates effectivity functions corresponding to veto functions.

Based on the observation that additive effectivity functions can be generated by positive probability measures on N and A, which can be regarded as *additive* TU-games on N and A, we introduce the following generalization of additive effectivity functions.

Let $v : 2^N \to [0,1]$ and $w : 2^A \to [0,1]$ be TU-games on N and A, which satisfy $v(N) = 1$ and $v(S) > 0$ for all $S \in \mathcal{P}_0(N)$, $w(A) = 1$ and $w(B) > 0$ for all $B \in \mathcal{P}_0(A)$. The games v and w generate an effectivity function $E(v,w) : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ as follows. For $S \in \mathcal{P}_0(N)$ and $B \in \mathcal{P}_0(A)$

$$B \in E(v,w)(S) \text{ if and only if } v(S) + w(B) > 1.$$

An effectivity function $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ is called *decomposable* if there exist TU-games v and w as above such that $E = E(v,w)$. For such TU-games v and w, $E(v,w)$ is called *the effectivity function generated by v and w.*

Here the TU-game v represents the veto power of coalitions and w represents the veto resistance of subsets of alternatives.

It readily follows from this definition that additive effectivity functions are decomposable. The following proposition illustrates that also effectivity functions corresponding to veto functions are decomposable.

Proposition 9.1.1 Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be an effectivity function. The following statements are equivalent

(i) E is decomposable

(ii) there exist $v : \mathcal{P}_0(N) \to [0,1]$ and $w : \mathcal{P}_0(A) \to [0,1]$ such that for all $S \in \mathcal{P}_0(N)$ and all $B \in \mathcal{P}_0(A)$ it holds that

$$B \in E(S) \text{ if and only if } v(S) + w(B) > 1$$

(iii) there exist $v_1 : \mathcal{P}_0(N) \to [0,1]$ and $w_1 : \mathcal{P}_0(A) \to [0,1]$ such that for all $S \in \mathcal{P}_0(N)$ and all $B \in \mathcal{P}_0(A)$ it holds that

$$B \in E(S) \text{ if and only if } v_1(S) + w_1(B) \geq 1$$

(iv) there exist $v_2 : \mathcal{P}_0(N) \to [0,1]$ and $w_2 : 2^A \to [0,1]$ with $w_2(\emptyset) := 0$ such that for all $S \in \mathcal{P}_0(N)$ and all $B \in \mathcal{P}_0(A)$ it holds that

$$B \in E(S) \text{ if and only if } v_2(S) \geq w_2(A \setminus B)$$

(v) there exist $v_3 : \mathcal{P}_0(N) \to [0,1]$ and $w_3 : 2^A \to [0,1]$ with $w_3(\emptyset) := 0$ such that for all $S \in \mathcal{P}_0(N)$ and all $B \in \mathcal{P}_0(A)$ it holds that

$$B \in E(S) \text{ if and only if } v_3(S) > w_3(A \setminus B).$$

As the proof of this proposition is quite technical but straightforward, it is omitted. From Proposition 9.1.1 (iv) we can derive the following corollary.

Corollary 9.1.2 Effectivity functions associated with monotonic simple games and effectivity functions corresponding to veto functions are decomposable.

9.2 Properties of TU-games and decomposable effectivity functions

In this section we examine relations between properties of the TU-games v and w and the effectivity function $E(v,w)$.

The following proposition shows that if v and w are monotonic, then $E(v,w)$ is N- and A-monotonic. The proof is straightforward.

Proposition 9.2.1 Let $E = E(v,w) : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be the decomposable effectivity function generated by the TU-games v and w. Then

(i) if v is monotonic, then E is N-monotonic

(ii) if w is monotonic, then E is A-monotonic.

With respect to the converse of this proposition it can be seen that if E is N-monotonic (A-monotonic) and decomposable, then there exist TU-games v and w with v (w) monotonic such that $E = E(v,w)$. (The TU-games v and w constructed in the proof of Theorem 9.3.4 are monotonic if E is monotonic.)
Proposition 9.2.2 shows that a decomposable effectivity function is convex if it can be generated by convex TU-games.

Proposition 9.2.2 Let $E = E(v,w) : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be the decomposable effectivity function generated by the TU-games v and w. If v and w are convex, then E is convex.

Proof. Let v be convex, i.e., for all $S, T \in \mathcal{P}_0(N)$: $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ (cf. Section 2.1), and let w be convex. Let $S, T \in \mathcal{P}_0(N)$, $B \in E(S)$ and $D \in E(T)$. We have to show that E is convex, i.e., $B \cap D \in E(S \cup T)$ or $B \cup D \in E(S \cap T)$. Since $v(S) + w(B) > 1$ and $v(T) + w(D) > 1$, we have

$$v(S) + v(T) + w(B) + w(D) > 2.$$

Using convexity of v and w now yields

$$v(S \cup T) + w(B \cap D) + v(S \cap T) + w(B \cup D) > 2.$$

Hence, $v(S \cup T) + w(B \cap D) > 1$ or $v(S \cap T) + w(B \cup D) > 1$. So, $B \cap D \in E(S \cup T)$ or $B \cup D \in E(S \cap T)$. □

The next example shows that $E(v,w)$ is not necessarily superadditive, if both v and w are superadditive.

Example 9.2.3 Let $N = \{1,2,3\}$ and $A = \{a,b,c\}$. Define $v : 2^N \to [0,1]$ by $v(\emptyset) = 0$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = 1/3$, $v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = 2/3$, and $v(N) = 1$, and define $w : 2^A \to [0,1]$ by $w(\emptyset) = 0$, $w(\{a\}) = w(\{b\}) = w(\{c\}) = 1/4$, $w(\{a,b\}) = w(\{a,c\}) = w(\{b,c\}) = 3/4$, and $w(A) = 1$. Then for all $S, T \in \mathcal{P}_0(N)$ with $S \cap T = \emptyset$ we have $v(S) + v(T) \leq v(S \cup T)$, and for all

$B, D \in \mathcal{P}_0(A)$ with $B \cap D = \emptyset$ we have $w(B) + w(D) \leq w(B \cup D)$. So v and w are superadditive. Furthermore, $\{a, b\} \in E(v, w)(\{1\})$ and $\{a, c\} \in E(v, w)(\{2\})$, but $\{a\} \notin E(v, w)(\{1, 2\})$. Hence, $E(v, w)$ is not superadditive.

It can be shown that $E(v, w)$ is superadditive whenever v is superadditive and w convex.

Theorem 9.2.4 states that if both v and w have a non-empty core, then also the core of $E(v, w)$ is non-empty for every preference profile.

Theorem 9.2.4 Let $E = E(v, w) : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be the decomposable effectivity function generated by the TU-games v and w. If v and w are balanced, then E is stable.

Proof. Let $x \in C(v)$ and $y \in C(w)$. Then $\sum_{i \in N} x_i = 1$ and $\sum_{a \in A} y_a = 1$. Furthermore, $x_i \geq v(\{i\}) > 0$ for all $i \in N$ and $y_a \geq w(\{a\}) > 0$ for all $a \in A$. So, the vectors x and y determine an additive effectivity function $E_{x,y}$. Moreover, $E(S) \subset E_{x,y}(S)$ for all $S \in \mathcal{P}_0(N)$, since $v(S) + w(B) > 1$ implies $\sum_{i \in S} x_i + \sum_{a \in B} y_a > 1$. Now stability of E follows directly from the fact that $E_{x,y}$ is stable. □

It is an open problem whether each stable decomposable effectivity function can be generated by TU-games v and w both having an non-empty core.

9.3 A characterization of decomposable effectivity functions

Moulin and Peleg (1982) showed that each neutral and A-monotonic effectivity function corresponds to a veto function and conversely. A characterization of additive effectivity functions is provided by Storcken (1994) using a property that strengthens convexity. In this section we will provide a characterization of the class of decomposable effectivity functions using a modification of the 'weak axiom of revealed preference' in social choice theory. This property is called the revealed power property.

Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be an effectivity function. Suppose that for all coalitions $S, T \in \mathcal{P}_0(N)$ and all subsets $B \in \mathcal{P}_0(A)$ of alternatives with $B \in E(S)$ and $B \notin E(T)$ we have, if $D \in \mathcal{P}_0(A)$ and $D \in E(T)$, then $D \in E(S)$. In this case we say that E satisfies the *revealed power property.*

The interpretation of this property is the following: If an effectivity function satisfies the revealed power property and a coalition S is effective for a certain subset of alternatives for which coalition T is not effective, then this 'reveals' that S has more power than T, i.e., S is effective for every subset that T is effective for.
It is clear that an effectivity function E satisfies the revealed power property if and only if for all $S, T \in \mathcal{P}_0(N)$ we have

$$E(S) \subset E(T) \text{ or } E(T) \subset E(S).$$

The following proposition shows that the revealed power property is a necessary condition to characterize decomposable effectivity functions.

Proposition 9.3.1 Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be an effectivity function. If E is decomposable, then E satisfies the revealed power property.

Proof. Let E be decomposable. Then there exist TU-games v and w such that $E = E(v, w)$. Let $S, T \in \mathcal{P}_0(N)$ with $E(S) \not\subset E(T)$. We show that $E(T) \subset E(S)$. Since there is a $B \in \mathcal{P}_0(A)$ with $v(S)+w(B) > 1$ and $v(T)+w(B) \leq 1$, it follows hat $v(S) > v(T)$. Now let $D \in E(T)$. Then $v(T)+w(D) > 1$ and hence $v(S)+w(D) > 1$, which implies that $D \in E(S)$. So we may conclude that $E(T) \subset E(S)$. □

It turns out that the revealed power property is also a sufficient condition to characterize decomposability. In order to prove this, we first introduce some additional notation.
Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be an effectivity function. The *dual of* E (Peleg (1984b)), $E^d : \mathcal{P}_0(A) \to 2^{\mathcal{P}_0(N)}$ is defined as follows. For $B \in \mathcal{P}_0(A)$

$$E^d(B) = \{S \in \mathcal{P}_0(N) \mid B \in E(S)\}.$$

(Note that E^d can be seen as an effectivity function for which the roles of N and A are interchanged.) We can also restate the revealed power property in terms of the dual of an effectivity function.

Lemma 9.3.2 Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be an effectivity function. Then E satisfies the revealed power property if and only if for all $B, D \in \mathcal{P}_0(A)$ we have $E^d(B) \subset E^d(D)$ or $E^d(D) \subset E^d(B)$.

Proof. We only prove the only if part. The other implication is similar.
Let E satisfy the revealed power property. Let $B, D \in \mathcal{P}_0(A)$ with $E^d(B) \not\subset E^d(D)$.

Then there exists a coalition $S \in \mathcal{P}_0(N)$ with $B \in E(S)$ and $D \notin E(S)$. Now let $T \in E^d(D)$. Then $D \in E(T)$, and since E satisfies the revealed power property, we have $E(S) \subset E(T)$. Since $B \in E(S)$, it follows that $B \in E(T)$, which implies $T \in E^d(B)$. Hence, $E^d(D) \subset E^d(B)$. □

In the following we use the equivalence relations $\sim_N$ on $\mathcal{P}_0(N)$ and $\sim_A$ on $\mathcal{P}_0(A)$, corresponding to an arbitrary effectivity function E, defined by

$$S \sim_N T \;\Leftrightarrow\; E(S) = E(T) \qquad \text{for all } S, T \in \mathcal{P}_0(N), \tag{9.1}$$

$$B \sim_A D \;\Leftrightarrow\; E^d(B) = E^d(D) \qquad \text{for all } B, D \in \mathcal{P}_0(A). \tag{9.2}$$

If E satisfies the revealed power property, it is possible to order the equivalence classes $[S_1], [S_2], \ldots, [S_k]$ induced by $\sim_N$ in a decreasing way, i.e.,

$$S \in [S_i],\ T \in [S_j],\ i < j \Rightarrow E(S) \supsetneq E(T). \tag{9.3}$$

(Note that $N \in [S_1]$).

By Lemma 9.3.2 it follows that if E satisfies the revealed power property, it is possible to order the equivalence classes $[B_1], [B_2], \ldots, [B_l]$ induced by $\sim_A$ such that

$$B \in [B_r],\ D \in [B_s],\ r < s \Rightarrow E^d(B) \supsetneq E^d(D). \tag{9.4}$$

(Note that $A \in [B_1]$).

Lemma 9.3.3 Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be an effectivity function which satisfies the revealed power property. Let $\sim_N$ and $\sim_A$ be the equivalence relations as defined in (9.1) and (9.2). Let the corresponding equivalence classes $[S_1], [S_2], \ldots, [S_k]$ and $[B_1], [B_2], \ldots, [B_l]$ be ordered as in (9.3) and (9.4), respectively. Then we have

(i) for all $i \in \{1, \ldots, k\}$ there exists an $s \in \{1, \ldots, l\}$ such that for all $S \in [S_i]$

$$E(S) = \bigcup_{r=1}^{s} [B_r]$$

(ii) for all $r \in \{1, \ldots, l\}$ there exists an $j \in \{1, \ldots, k\}$ such that for all $B \in [B_r]$

$$E^d(B) = \bigcup_{i=1}^{j} [S_i]$$

(iii) $k = l$

(iv) for all $i \in \{1, \ldots, k\}$ and $S \in [S_i]$

$$E(S) = \bigcup_{r=1}^{k+1-i} [B_r].$$

Proof. (i) Let $i \in \{1, \ldots, k\}$ and $S \in [S_i]$. It suffices to show that for $t \in \{1, \ldots, l\}$, for $B \in [B_t]$ with $B \in E(S)$, and for $D \in [B_r]$ with $1 \leq r \leq t$, we have $D \in E(S)$. This follows immediately from the fact that $E^d(B) \subset E^d(D)$.

(ii) Similar to (i).

(iii) From (i) we derive that $l \geq k$ and from (ii) it follows that $k \geq l$. Hence, $k = l$.

(iv) Follows immediately from (i) and (iii). □

Now we are able to prove

Theorem 9.3.4 Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be an effectivity function. Then E is decomposable if and only if it satisfies the revealed power property.

Proof. The only if part follows from Proposition 9.3.1. To prove the if part, let E satisfy the revealed power property. Let $[S_1], [S_2], \ldots, [S_k]$ be the equivalence classes corresponding to $\sim_N$ ordered as in (9.3), and let $[B_1], [B_2], \ldots, [B_l]$ be the equivalence classes corresponding to $\sim_A$ ordered as in (9.4). By Lemma 9.3.3 we have $k = l$ and for all $S \in [S_i]$ it holds that

$$E(S) = \bigcup_{r=1}^{k+1-i} [B_r].$$

Now define TU-games $v : 2^N \to [0,1]$ and $w : 2^A \to [0,1]$ as follows.
$v(\emptyset) := 0$, $w(\emptyset) := 0$, and

$$\begin{aligned} v(S) &:= (k+1-i)/k \qquad \text{for all } S \in [S_i] \text{ and } i \in \{1, \ldots, k\}, \\ w(B) &:= (k+1-r)/k \qquad \text{for all } B \in [B_r] \text{ and } r \in \{1, \ldots, k\}. \end{aligned}$$

Let $S \in [S_i]$ and $B \in [B_r]$. Then

$$\begin{aligned} v(S) + w(B) > 1 &\Leftrightarrow (k+1-i)/k + (k+1-r)/k > 1 \\ &\Leftrightarrow k+2-i > r \\ &\Leftrightarrow r \leq k+1-i \\ &\Leftrightarrow B \in E(S). \end{aligned}$$

Hence $E = E(v, w)$, which completes the proof. □

9.4 Decomposability and echelon matrices

In Section 8.2 we have seen that an effectivity function E can be represented by a $\{0,1\}$-matrix I^E of size $2^n - 1$ by $2^{|A|} - 1$, where for $S \in \mathcal{P}_0(N)$ and $B \in \mathcal{P}_0(A)$,

$$I^E(S,B) = 1 \text{ if and only if } B \in E(S).$$

In this section we provide a characterization of decomposable effectivity functions in terms of matrices. We show that an effectivity function is decomposable if and only if it can be represented by a $\{0,1\}$-matrix in echelon form in which the 1's are 'separated' from the 0's. (see Figure 9.1)

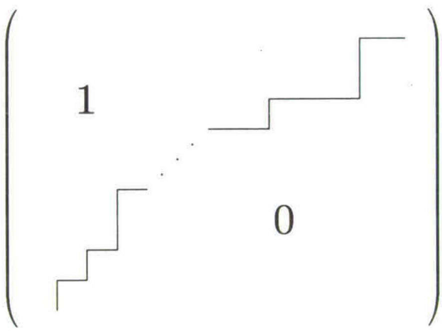

Figure 9.1.

Theorem 9.4.1 Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be an effectivity function and I^E a matrix that represents E. Then E is decomposable if and only if it is possible to rearrange the rows and columns of I^E in such a way that the rearranged matrix has an echelon form as in Figure 9.1.

Proof. Let E be decomposable. By Theorem 9.3.4, E satisfies the revealed power property. Define a relation $Q \subset \mathcal{P}_0(N) \times \mathcal{P}_0(N)$ by $T\ Q\ S$ if and only if $E(T) \supset E(S)$ for all $S, T \in \mathcal{P}_0(N)$. Then Q is transitive and since E satisfies the revealed power property, Q is also complete. Rearrange the rows of I^E in a decreasing order according to Q, say, in the order $S_1, \ldots, S_{2^n-1}$.

Consider the column corresponding to a subset $B \in \mathcal{P}_0(A)$ of alternatives. Define $r(B) := \max\{r \in \{1, \ldots, 2^n - 1\} \mid B \in E(S_r)\}$ and take $r \leq r(B)$. Since, $B \in E(S_{r(B)})$ and $E(S_{r(B)}) \subset E(S_r)$, we have $I^E(S_r, B) = 1$, from which it immediately follows that every column of the rearranged matrix has the form $(1, \ldots, 1, 0, \ldots, 0)^T$

(x^T denotes the transposed of a vector x). Then obviously, by reordering the columns of this matrix, we can obtain a matrix in an echelon form.

To prove the if part, suppose it is possible to rearrange the rows and columns of I^E in such a way that we obtain a matrix in the form of Figure 9.1. Suppose the columns of this matrix are arranged in the order $B_1, \ldots, B_{2^{|A|-1}}$. Let $S \in \mathcal{P}_0(N)$. Define $m(S) := \max\{r \in \{1, \ldots, 2^{|A|-1}\} \mid I^E(S, B_r) = 1\}$. Since the row corresponding to S has the form $(1, \ldots, 1, 0, \ldots, 0)$, it follows that $E(S) = \{B_1, \ldots, B_{m(S)}\}$. From this observation it immediately follows that for $S, T \in \mathcal{P}_0(N)$ we have $E(S) \subset E(T)$ if and only if $m(S) \leq m(T)$. Hence, E satisfies the revealed power property and hence, by Theorem 9.3.4, E is decomposable. $\square$

Chapter 10

Effectivity functions and claim game correspondences

In Section 8.5 we have studied relations between non-cooperative game forms and effectivity functions which describe coalitional power in a society. In fact, this line of research of establishing connections between cooperative and non-cooperative types of games was initiated by von Neumann and Morgenstern (1994), who showed that it is possible to construct a superadditive TU-game from a game in strategic form, and conversely. Later, further relationships between cooperative and non-cooperative games were established (see for example Nash (1950) and Aumann (1961,1967)).
Borm and Tijs (1992) introduced a 'claim' game in strategic form corresponding to an NTU-game. In the claim game, strategies of players can be interpreted as claims on coalitions and payoffs. Among others things Borm and Tijs showed that if the NTU-game is superadditive, strong core elements of the NTU-game correspond to strong Nash equilibria of the associated claim game.
In this chapter, which is based on Otten, Borm, Storcken and Tijs (1995), we use techniques similar to those of Borm and Tijs (1992) to examine relations between effectivity functions and game correspondences. Game correspondences were introduced in Peleg (1984b) as an extension of game forms. Whereas in a game form the outcome function assigns to every strategy vector one alternative, in a game correspondence the outcome function assigns to each strategy vector a subset of alternatives.
In Section 10.1 the formal definition of a game correspondence is given, and, given a preference profile, we define strong Nash equilibria of game correspondences. Fur-

thermore, we define the α- and β-effectivity functions associated with a game correspondence as in Peleg (1984b).

In Section 10.2 we extend the definition of the core (at a given preference profile) to be a collection of subsets of alternatives rather than only one subset. We show that this extended core, which we call the setcore of an effectivity function, is never empty if the effectivity function is upper cycle free.

Section 10.3 is the central part of this chapter. We construct the claim game correspondence $G(E)$ associated with an upper cycle free effectivity function E and show that this game correspondence is tight. Furthermore, we present a result similar to Theorem 8.5.2, which illustrates that the class of effectivity functions associated with tight game correspondences is strictly larger than the class of effectivity functions that can be derived from a tight game form. Moreover, Section 10.3 shows that the setcore of a superadditive effectivity function (at a certain profile) exactly corresponds to the outcomes of strong Nash equilibria of the associated claim game correspondence (at that profile). A combination of these results leads to an analogue of Theorem 8.5.3 for a larger class of effectivity functions.

Finally, Section 10.4 discusses a process for deriving a game form from a claim game correspondence in such a way that the α-effectivity function does not change. As a result of this process we obtain an alternative proof of Theorem 8.5.1.

10.1 Game correspondences

In this section we define game correspondences and their associated α- and β-effectivity functions. Moreover, we extend preferences over a finite set of alternatives A to preferences over $\mathcal{P}_0(A)$ in order to define strong Nash equilibria for game correspondences.

A *game correspondence* (Peleg (1984b)) is an (n+2)-tuple $G = (X_1, \ldots, X_n, A, \pi)$ where X_i is a non-empty set of strategies for each $i \in N$, A is a finite set of alternatives, and $\pi : X_N \to \mathcal{P}_0(A)$ is non-imposed, i.e., for each $a \in A$ there is a strategy $\sigma_N \in X_N$ such that $\pi(\sigma_N) = \{a\}$.

The interpretation of G is similar to the interpretation of a game form: Each agent $i \in N$ chooses a strategy $\sigma_i \in X_i$, and then the outcome correspondence π determines a non-empty subset $\pi(\sigma_N)$ of alternatives.

Peleg (1984b) extended the definition of the α- and β-effectivity function associated with a game form to game correspondences.
Let $G = (X_1, \ldots, X_n, A, \pi)$ be a game correspondence. The *α- and β-effectivity functions E_α^G and E_β^G associated with* G are defined as follows. Let $S \in \mathcal{P}_0(N)$. Then

$$E_\alpha^G(S) := \{B \in \mathcal{P}_0(A) \mid \exists \sigma_S \in X_S\ \forall \tau_{N\setminus S} \in X_{N\setminus S} :\ \pi(\sigma_S, \tau_{N\setminus S}) \subset B\}$$

$$E_\beta^G(S) := \{B \in \mathcal{P}_0(A) \mid \forall \tau_{N\setminus S} \in X_{N\setminus S}\ \exists \sigma_S \in X_S :\ \pi(\sigma_S, \tau_{N\setminus S}) \subset B\}.$$

The reader can easily verify that E_α^G and E_β^G are indeed effectivity functions (since π is non-imposed) and that $E_\alpha^G(S) \subset E_\beta^G(S)$ for all $S \in \mathcal{P}_0(N)$ and all game correspondences G. A game correspondence G is called *tight* if $E_\alpha^G = E_\beta^G$. Furthermore, it is easy to see that for every game correspondence G, E_α^G and E_β^G are both N-monotonic and A-monotonic and that E_α^G is superadditive. However contrary to game forms, E_β^G need not be maximal.

Since the outcome function of a game correspondence is a function to $\mathcal{P}_0(A)$ instead of a function to A, we first must extend preference profiles on A to preference profiles on $\mathcal{P}_0(A)$ in order to be able to define the concept of a strong Nash equilibrium in the context of game correspondences.
Let $R \in \mathcal{R}$ be a preference relation on A and let P be the asymmetric part of R. We define the extension $\widetilde{P}$ of P to $\mathcal{P}_0(A)$ as follows. For all $B, B' \in \mathcal{P}_0(A)$ we have $B'\ \widetilde{P}\ B$ if and only if $B \setminus B' \neq \emptyset$ and

(i) for all $b' \in B' \setminus B$ and all $b \in B$ we have $b'\ P\ b$,

(ii) for all $b' \in B'$ and all $b \in B \setminus B'$ we have $b'\ P\ b$.

We write $\widetilde{P}_S$ instead of $(\widetilde{P}_i)_{i\in S}$.
It readily follows that, if we restrict $\widetilde{P}$ to the singletons of $\mathcal{P}_0(A)$, then this restriction can be identified with P by identifying a singleton with its unique element. Note that $\widetilde{P}$ is also transitive. According to this definition $B' \in \mathcal{P}_0(A)$ is (strictly) preferred to $B \in \mathcal{P}_0(A)$ if, in going from B to B', the elements being added to B (i.e., $B' \setminus B$) are better than the ones already present, and the elements of B being dropped (i.e., $B \setminus B'$) are worse than the elements of B which are kept (i.e., $B \cap B'$).
Note that $B'\ \widetilde{P}\ B$ implies that B' is not a set that strictly contains B.

Now we are able to give the definition of a strong Nash equilibrium for game correspondences. Let $G = (X_1, \ldots, X_n, A, \pi)$ be a game correspondence and $R_N \in \mathcal{R}_N$ a profile. A strategy vector $\sigma_N \in X_N$ is called a *strong Nash equilibrium of* G *at* R_N if there do not exist $S \in \mathcal{P}_0(N)$ and $\tau_S \in X_S$ such that $\pi(\sigma_{N \setminus S}, \tau_S)\ \tilde{P}_S\ \pi(\sigma_N)$. A game correspondence G is called *strongly consistent* if the set of strong Nash equilibria of G at R_N, denoted by $SNE(G, R_N)$, is non-empty for all $R_N \in \mathcal{R}_N$.

10.2 The setcore of an effectivity function

In Section 8.3 we defined the core of an effectivity function at a profile as that subset of the set of alternatives which consists of all undominated elements with respect to the profile. In order to obtain relations between strong Nash equilibria of game correspondences and 'core elements' of effectivity functions, we modify the notion of the core in the sense that it will assign to an effectivity function and a preference profile a collection of subsets of A rather than one subset. This modification of the core is called the setcore.

Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be an effectivity function and let R_N be a profile on A. Let $B \in \mathcal{P}_0(A)$ be a subset of alternatives. B is an element of the *setcore of* E *at* R_N, $Setcore(E, R_N)$, if there do not exist $S \in \mathcal{P}_0(N)$ and $B' \in E(S)$ such that $B'\ \tilde{P}_S\ B$. The setcore of E at R_N definitionally extends the core to subsets of alternatives. One easily checks that $a \in Core(E, R_N)$ if and only if $\{a\} \in Setcore(E, R_N)$. Moreover, it holds that, if $B \subset Core(E, R_N)$, $B \neq \emptyset$, then $B \in Setcore(E, R_N)$.
For, suppose $B \notin Setcore(E, R_N)$. Then there exist a coalition $S \in \mathcal{P}_0(N)$ and a $B' \in E(S)$ such that $B'\ \tilde{P}_S\ B$. Hence, $B \setminus B' \neq \emptyset$. Take $a \in B \setminus B'$. Since $B'\ \tilde{P}_S\ B$, we obtain by definition of $\tilde{P}_S$ that $b'\ P_S\ a$ for all $b' \in B'$. Hence $a \notin Core(E, R_N)$, which leads to a contradiction.

From the previous remark it follows that the setcore of an effectivity function (at a profile) is non-empty, whenever the core of the effectivity function (at this profile) is non-empty. However, the next example shows that if the core of an effectivity function at a profile is empty, then the setcore is not necessarily empty.

Example 10.2.1 Let $N = \{1, 2, 3\}$ and $A = \{a, b, c\}$. Define $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ as follows. For $S \in \mathcal{P}_0(N)$

$$E(S) := \begin{cases} \{A\} & \text{if } |S| = 1 \\ \mathcal{P}_0(A) & \text{if } |S| > 1. \end{cases}$$

E is the effectivity function associated with the simple game in which any majority is winning (cf. Section 8.4).
Define the preference profile $R_N = (R_1, R_2, R_3)$ on A by

$R_1 = a\ b\ c,$

$R_2 = b\ c\ a,$

$R_3 = c\ a\ b.$

Here there are no indifferences and the preferences of the players are denoted in decreasing order, so player 1 likes a the most, then b, and then c, and so on.
One easily checks that $Core(E, R_N) = \emptyset$. However, the setcore of E at R_N is non-empty: $Setcore(E, R_N) = \{A\}$, which seems a rather natural solution.

Now we formulate the main theorem of this section, which yields an existence theorem of the setcore on the class of upper cycle free effectivity functions.

Theorem 10.2.2 Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be an upper cycle free effectivity function, and let $R_N \in \mathcal{R}_N$ be a profile on A. Then $Setcore(E, R_N) \neq \emptyset$.

Proof. The proof is by induction on $|A|$.
Clearly, if $|A| = 1$, then $Setcore(E, R_N) \neq \emptyset$.
Let $|A| > 1$, and assume that $Setcore(E', R'_N) \neq \emptyset$ for all upper cycle free effectivity functions $E' : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A')}$ with $|A'| < |A|$ and all profiles R'_N on A'.
Suppose $Setcore(E, R_N) = \emptyset$. Then, in particular, there exist a coalition $T \in \mathcal{P}_0(N)$ and an $A' \in E(T)$ such that $A'\ \widetilde{P}_T\ A$. By definition of $\widetilde{P}_T$ it follows that $|A'| < |A|$ and that $A'\ \widetilde{P}_T\ (A \setminus A')$.
Define $E' : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A')}$ by $E'(N) := \mathcal{P}_0(A')$ and for $S \in \mathcal{P}_0(N) \setminus \{N\}$ and $B \in \mathcal{P}_0(A')$

$$B \in E'(S) \ \Leftrightarrow\ B = A' \text{ or there exist } C \subset A \setminus A' \text{ and } D \subset A', D \neq \emptyset, \text{ such that } B \cup C \in E(S) \text{ and } (B \cup C)\ \widetilde{P}_S\ D.$$

Clearly, E' is an effectivity function. Furthermore, let R'_N (P'_N) be the restriction of R_N (P_N) to A'. We will show that

(I) $Setcore(E', R'_N) = \emptyset$

(II) E' is upper cycle free.

(I) Let $B' \in \mathcal{P}_0(A')$. Since $Setcore(E, R_N) = \emptyset$, there are $S \in \mathcal{P}_0(N)$ and $B \in E(S)$ such that $B\ \widetilde{P}_S\ B'$. Let $X := B \cap A'$.

Claim: $X \in E'(S)$ and $X\ \widetilde{P}'_S\ B'$.

From the claim it immediately follows that $B' \notin Setcore(E', R'_N)$.

Hence, $Setcore(E', R'_N) = \emptyset$.

Proof of the claim: First we show that $X \neq \emptyset$. Suppose $X = \emptyset$. Then $B \subset A \setminus A'$ and therefore, $B' \cap B = \emptyset$. Since $A'\ \widetilde{P}_T\ (A \setminus A')$, it follows that $S \cap T = \emptyset$. Since $A' \cap B = \emptyset$, this leads to a contradiction with upper cycle freeness of E. Hence, $X \neq \emptyset$.

Since $X \cup (B \setminus A') = B \in E(S)$ and $B\ \widetilde{P}_S\ B'$, it follows that $X \in E'(S)$.

Using the fact that $B\ \widetilde{P}_S\ B'$ and $B' \cap B \subset X \subset B$, it follows from the definition of $\widetilde{P}'_S$ that $X\ \widetilde{P}'_S\ B'$.

(II) Let $S_1, \ldots, S_k \in \mathcal{P}_0(N)$ with $S_r \cap S_t = \emptyset$ for all $r, t \in \{1, \ldots, k\}$, $r \neq t$, and let $B_1, \ldots, B_k \in \mathcal{P}_0(A')$ be such that $B_r \in E'(S_r)$ for all $r \in \{1, \ldots, k\}$. It suffices to prove that $\bigcap_{r=1}^k B_r \neq \emptyset$.

Suppose that $\bigcap_{r=1}^k B_r = \emptyset$. We assume, without loss of generality, that $B_r \neq A'$ for all r. From the definition of E' it follows that there are $C_1, \ldots, C_k \subset A \setminus A'$ and $D_1, \ldots, D_k \subset A'$ such that $B_r \cup C_r \in E(S_r)$ and $(B_r \cup C_r)\ \widetilde{P}_{S_r}\ D_r$ for all r. Since E is upper cycle free, it follows that $\bigcap_{r=1}^k (B_r \cup C_r) = \bigcap_{r=1}^k C_r \neq \emptyset$. Let $b \in \bigcap_{r=1}^k C_r$. Since $b \notin A'$ and $(B_r \cup C_r)\ \widetilde{P}_{S_r}\ D_r$ for all r, it follows that $\{b\}\ \widetilde{P}_{S_r}\ D_r$ for all r. As $A'\ \widetilde{P}_T\ (A \setminus A')$, we have $T \cap S_r = \emptyset$ for all r.

Consider $T, S_1, \ldots, S_k$ and $A' \in E(T), B_1 \cup C_1 \in E(S_1), \ldots, B_k \cup C_k \in E(S_k)$. Since $\bigcap_{r=1}^k (B_r \cup C_r) \cap A' = \emptyset$, it follows that E is not upper cycle free. This leads to a contradiction and hence E' is upper cycle free.

Statements (I) and (II) are in contradiction with the induction hypothesis.

So, $Setcore(E, R_N) \neq \emptyset$. □

The next example illustrates that upper cycle freeness is not a necessary property to guarantee non-emptiness of the setcore at every profile.

Example 10.2.3 Let $N = \{1, 2, 3\}$ and $A = \{a_0, \ldots, a_5\}$. Let $B_1 = \{a_0, a_1, a_3\}$, $B_2 = \{a_1, a_2, a_4\}$, $B_3 = \{a_3, a_4, a_5\}$.

Define the effectivity function E by

$$
\begin{aligned}
&E(\{i\}) = \{B_i, A\} && i = 1, 2, 3\\
&E(S) = \{A\} && \text{if } |S| = 2\\
&E(N) = \mathcal{P}_0(A).
\end{aligned}
$$

Clearly, E is not upper cycle free.
We leave it to the reader to verify that $Setcore(E, R_N) \neq \emptyset$ for all profiles R_N.

10.3 Claim game correspondences

This section shows how for an upper cycle free effectivity function E, one can construct a game correspondence $G(E)$ such that $G(E)$ is tight and $E \subset E_\alpha^{G(E)}$. The game correspondence $G(E)$ is called the claim game correspondence associated with E. We provide necessary and sufficient conditions on E such that $E = E_\alpha^{G(E)}$. Finally, this section establishes relationships between the setcore of E at a preference profile R_N and the set of strong Nash equilibria of the claim game correspondence $G(E)$ at R_N.

Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be an upper cycle free effectivity function. The *claim game correspondence* $G(E)$ *associated with* E is given by $G(E) = (X_1, \dots, X_n, A, \pi)$, where for each $i \in N$

$$X_i := \{(S, B) \in \mathcal{P}_0(N) \times \mathcal{P}_0(A) \mid i \in S,\ B \in E(S)\},$$

and for $\sigma_N = (S_i, B_i)_{i \in N} \in X_N$, $\pi(\sigma_N)$ is defined by

$$\pi(\sigma_N) := \begin{cases} \bigcap\{B \in \mathcal{P}_0(A) \mid B \in F(\sigma_N)\} & \text{if } F(\sigma_N) \neq \emptyset \\ A & \text{if } F(\sigma_N) = \emptyset. \end{cases}$$

Here $F : X_N \to \mathcal{P}_0(A)$ is defined as follows. For $\sigma_N \in X_N$

$$F(\sigma_N) := \{B \in \mathcal{P}_0(A) \mid \exists S \in \mathcal{P}_0(N) : B \in E(S) \text{ and } \forall i \in S,\ \sigma_i = (S, B)\}.$$

Note that $\pi(\sigma_N) \neq \emptyset$ for all $\sigma_N \in X_N$ because E is upper cycle free.
In the claim game correspondence $G(E)$ the strategy (S_i, B_i) of player $i \in N$ can be interpreted as a claim in the following way. Player i wants to form coalition S_i he belongs to and he wants the final outcome to be in a subset B_i of alternatives for which S_i is effective. According to the outcome function π, the final outcome will certainly be in B_i if all the players in S_i have exactly the same claims. The idea behind this definition is similar to the construction of Borm and Tijs (1992).

Example 10.3.1 Let $N = \{1, 2, 3, 4\}$ and $A = \{a_0, a_1, a_2, a_3, a_4, a_5, a_6\}$.
Define $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ by
$E(\{1\}) = \{\ \{a_0, a_4\},\ \{a_1, a_2, a_4\},\ A\}$, $E(\{1, 2\}) = \{\ \{a_1, a_2, a_5, a_6\},\ A\}$,

$E(\{2,3\}) = \{\ \{a_0, a_1, a_2, a_5\},\ A\}$, $E(\{3,4\}) = \{\ \{a_1, a_3, a_4, a_6\},\ A\}$,
$E(N) = \mathcal{P}_0(A)$, and $E(S) = \{A\}$ else.
Then E is upper cycle free, so $G(E) = (X_1, \dots, X_n, A, \pi)$ is well-defined.
Note that E is not superadditive. Define the strategy $\sigma_N \in X_N$ by $\sigma_1 = (\{1\}, \{a_0, a_4\})$, $\sigma_2 = \sigma_3 = (\{2,3\}, \{a_0, a_1, a_2, a_5\})$, and $\sigma_4 = (\{3,4\}, \{a_1, a_3, a_4, a_6\})$.
Then $F(\sigma_N) = \{\ \{a_0, a_4\},\ \{a_0, a_1, a_2, a_5\}\}$ and therefore,
$\pi(\sigma_N) = \{a_0, a_4\} \cap \{a_0, a_1, a_2, a_5\} = \{a_0\}$.

Tightness of claim game correspondences associated with upper cycle free effectivity functions follows from Proposition 10.3.2.

Proposition 10.3.2 Let the effectivity function $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be upper cycle free. Then

$$E \subset E_\alpha^{G(E)} = E_\beta^{G(E)}.$$

Proof. Let $S \in \mathcal{P}_0(N)$ and $B \in E(S)$. Define $\sigma_S \in X_S$ by $\sigma_i := (S, B)$ for all $i \in S$. Then for all $\tau_{N\setminus S} \in X_{N\setminus S}$, we have $B \in F(\sigma_S, \tau_{N\setminus S})$. Hence, $\pi(\sigma_S, \tau_{N\setminus S}) \subset B$ for all $\tau_{N\setminus S} \in X_{N\setminus S}$. So, $B \in E_\alpha^{G(E)}(S)$.
To prove tightness of $G(E)$ it suffices to show that $E_\beta^{G(E)}(S) \subset E_\alpha^{G(E)}(S)$ for all $S \in \mathcal{P}_0(N)$. Let $S \in \mathcal{P}_0(N)$ and $B \in E_\beta^{G(E)}(S)$. Define $\widehat{\tau}_{N\setminus S} \in X_{N\setminus S}$ by $\widehat{\tau}_i := (\{i\}, A)$ for all $i \in N \setminus S$. Since $B \in E_\beta^{G(E)}(S)$, there is a $\sigma_S \in X_S$ such that $\pi(\sigma_S, \widehat{\tau}_{N\setminus S}) \subset B$. Hence, $\bigcap\{D \in \mathcal{P}_0(A) \mid D \in F(\sigma_S, \widehat{\tau}_{N\setminus S})\} \subset B$. Because $(F(\sigma_S, \widehat{\tau}_{N\setminus S}) \setminus \{A\}) \subset F(\sigma_S, \tau_{N\setminus S})$ for all $\tau_{N\setminus S} \in X_{N\setminus S}$, we obtain $\bigcap\{D \in \mathcal{P}_0(A) \mid D \in F(\sigma_S, \tau_{N\setminus S})\} \subset B$ for all $\tau_{N\setminus S} \in X_{N\setminus S}$. Hence, $B \in E_\alpha^{G(E)}(S)$. □

The next proposition illustrates the importance of the condition of upper cycle freeness.

Proposition 10.3.3 Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be an effectivity function. There exists a game correspondence G such that $E \subset E_\alpha^G$ if and only if E is upper cycle free.

Proof. $(\Rightarrow)$ Since E_α^G is superadditive this follows immediately from Lemma 8.2.2.
$(\Leftarrow)$ This follows immediately from Proposition 10.3.2. □

Theorem 10.3.6 below characterizes the properties an effectivity function E must satisfy for coincidence of E and $E_\alpha^{G(E)}$. For this we need the monotonicity result of Lemma 10.3.4.

Lemma 10.3.4 Let E and E' be upper cycle free effectivity functions and let $G(E) = (X_1, \ldots, X_n, A, \pi)$ and $G(E') = (X'_1, \ldots, X'_n, A, \pi')$ be the associated claim game correspondences. If $E \subset E'$ then $E_\alpha^{G(E)} \subset E_\alpha^{G(E')}$.

Proof. Let $B \in E_\alpha^{G(E)}(S)$ for some $S \in \mathcal{P}_0(N)$. If $B = A$, then $B \in E_\alpha^{G(E')}(S)$. Suppose $B \neq A$. There is a strategy $\sigma_S = (S_i, B_i)_{i \in S} \in X_S$ such that $\pi(\sigma_S, \tau_{N \setminus S}) \subset B$ for all $\tau_{N \setminus S} \in X_{N \setminus S}$. Take $\tau_{N \setminus S} \in X_{N \setminus S}$ such that $\tau_i = (\{i\}, A)$ for all $i \in X_{N \setminus S}$. Let $S_0 := \{i \in S \mid \exists j \in S_i [S_j \neq S_i \text{ or } B_j \neq B_i]\}$. Since $\pi(\sigma_S, \tau_{N \setminus S}) \subset B \neq A$, we have $S_0 \neq S$ and there is a partition $\{S_1^*, \ldots, S_k^*\}$ of $S \setminus S_0$ and $B_r^* \in E(S_r^*)$ for all $r \in \{1, \ldots, k\}$ such that $(S_i, B_i) = (S_r^*, B_r^*)$ for all $i \in S_r^*$ and all $r \in \{1, \ldots, k\}$ and $\bigcap_{r=1}^k B_r^* \subset B$. Since $\sigma_S \in X'_S$, and $B_r^* \in F(\sigma_S, \tau'_{N \setminus S})$ for $r \in \{1, \ldots, k\}$ and all $\tau'_{N \setminus S} \in X'_{N \setminus S}$, we obtain $\pi'(\sigma_S, \tau'_{N \setminus S}) \subset B$ for all $\tau'_{N \setminus S} \in X'_{N \setminus S}$. So $B \in E_\alpha^{G(E')}(S)$. □

In Section 8.2 we defined the superadditive cover $\overline{E}$ of an effectivity function E. The following proposition states that A-monotonicity is a sufficient condition for coincidence of $\overline{E}$ and $E_\alpha^{G(E)}$.

Proposition 10.3.5 Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be an upper cycle free effectivity function. Then

(i) $E_\beta^{G(E)} = E_\alpha^{G(E)} = E_\alpha^{G(\overline{E})} = E_\beta^{G(\overline{E})}$

(ii) $\overline{E} = E_\alpha^{G(\overline{E})}$ if and only if $\overline{E}$ is A-monotonic

Proof. (i) Because of Proposition 10.3.2 it suffices to prove that $E_\alpha^{G(E)} = E_\alpha^{G(\overline{E})}$.
($\subset$) This follows by Lemma 10.3.4.
($\supset$) Let $G(E) := (X_1, \ldots, X_n, A, \pi)$ and $G(\overline{E}) := (\overline{X}_1, \ldots, \overline{X}_n, A, \overline{\pi})$.
Let $S \in \mathcal{P}_0(N)$ and $B \in E_\alpha^{G(\overline{E})}(S)$. If $B = A$, then $B \in E_\alpha^{G(E)}(S)$. Suppose $B \neq A$. Then there is a $\overline{\sigma}_S \in \overline{X}_S$ such that $\overline{\pi}(\overline{\sigma}_S, \overline{\tau}_{N \setminus S}) \subset B$ for all $\overline{\tau}_{N \setminus S} \in \overline{X}_{N \setminus S}$. Take $\overline{\tau}_{N \setminus S} \in \overline{X}_{N \setminus S}$ such that $\overline{\tau}_i = (\{i\}, A)$ for all $i \in N \setminus S$. As in the proof of Lemma 10.3.4, since $\overline{\pi}(\overline{\sigma}_S, \overline{\tau}_{N \setminus S}) \subset B$ one can find disjoint subcoalitions $S_1, \ldots, S_k$ of S and $B_r \in \overline{E}(S_r)$ for all $r \in \{1, \ldots, k\}$ such that $\overline{\sigma}_i = (S_r, B_r)$ for all $i \in S_r$ and $r \in \{1, \ldots, k\}$, and $\bigcap_{r=1}^k B_r \subset B$. For all $r \in \{1, \ldots, k\}$, by definition of $\overline{E}$, there are partitions $\{S_{r1}, \ldots, S_{rk_r}\}$ of S_r and there are $B_{rs} \in E(S_{rs})$ for all $s \in \{1, \ldots, k_r\}$ such that $B_r = \bigcap_{s=1}^{k_r} B_{rs}$.
Define $\sigma_S \in X_S$ by $\sigma_i := (S_{rs}, B_{rs})$ for all $i \in S_{rs}$, $s \in \{1, \ldots, k_r\}$, and $r \in \{1, \ldots, k\}$. Then $B_{rs} \in F(\sigma_S, \tau_{N \setminus S})$ for all $s \in \{1, \ldots, k_r\}$, $r \in \{1, \ldots, k\}$, and $\tau_{N \setminus S} \in X_{N \setminus S}$.

Since $B \supset \bigcap_{r=1}^{k} B_r = \bigcap_{r=1}^{k} \bigcap_{s=1}^{k_r} B_{rs}$ we obtain $\pi(\sigma_S, \tau_{N\setminus S}) \subset B$ for all $\tau_{N\setminus S} \in X_{N\setminus S}$. Hence, $B \in E_\alpha^{G(E)}(S)$.

(ii) ($\Leftarrow$) Suppose $\overline{E}$ is A-monotonic. By Proposition 10.3.2 it is sufficient to prove that $\overline{E} \supset E_\alpha^{G(\overline{E})}$. Therefore, let $S \in \mathcal{P}_0(N)$ and $B \in E_\alpha^{G(\overline{E})}(S)$. If $B = A$, then $B \in \overline{E}(S)$. Suppose $B \neq A$. Then there exists a $\overline{\sigma}_S \in \overline{X}_S$ such that $\overline{\pi}(\overline{\sigma}_S, \overline{\tau}_{N\setminus S}) \subset B$ for all $\overline{\tau}_{N\setminus S} \in \overline{X}_{N\setminus S}$. Take $\overline{\tau}_{N\setminus S} \in \overline{X}_{N\setminus S}$ such that $\overline{\tau}_i = (\{i\}, A)$ for all $i \in N \setminus S$. Since $\overline{\pi}(\overline{\sigma}_S, \overline{\tau}_{N\setminus S}) \subset B$, one can find disjoint subsets $S_1, \ldots, S_k$ of S and $B_r \in \overline{E}(S_r)$ for all $r \in \{1, \ldots, k\}$ such that $\overline{\pi}(\overline{\sigma}_S, \overline{\tau}_{N\setminus S}) = \bigcap_{r=1}^{k} B_r$. Since $\overline{E}$ is superadditive, we obtain $\bigcap_{r=1}^{k} B_r \in \overline{E}(\bigcup_{r=1}^{k} S_r)$. Since $\overline{E}$ is superadditive and hence N-monotonic, this implies $\bigcap_{r=1}^{k} B_r \in \overline{E}(S)$. Hence, since $\overline{E}$ is A-monotonic and $\bigcap_{r=1}^{k} B_r \subset B$ it follows that $B \in \overline{E}(S)$.

($\Rightarrow$) Suppose $\overline{E} = E_\alpha^{G(\overline{E})}$. Then $\overline{E}$ is A-monotonic since $E_\alpha^{G(\overline{E})}$ is A-monotonic. □

Using Proposition 10.3.5 we are now able to prove the following result.

Theorem 10.3.6 Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be an effectivity function. Then

$$E = E_\alpha^{G(E)} = E_\beta^{G(E)} \text{ if and only if } E \text{ is superadditive and } A\text{-monotonic.}$$

Proof. ($\Rightarrow$) Clearly, if $E = E_\alpha^{G(E)} = E_\beta^{G(E)}$, then E is superadditive and A-monotonic, since $E_\alpha^{G(E)}$ is superadditive and A-monotonic.

($\Leftarrow$) Let E be superadditive and A-monotonic. Then $E = \overline{E}$ by Lemma 8.2.3. Using A-monotonicity Proposition 10.3.5 implies that $E = E_\alpha^{G(E)} = E_\beta^{G(E)}$. □

As a result of Theorem 10.3.6 we obtain the following counterpart of Theorem 8.5.2.

Corollary 10.3.7 Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be an effectivity function. Then there exists a tight game correspondence G such that $E = E_\alpha^G = E_\beta^G$ if and only if E is A-monotonic and superadditive.

In the last part of this section we examine relations between solution concepts of effectivity functions at a profile R_N and solution concepts of the associated claim game correspondence at R_N. It is shown that, if the effectivity function is superadditive, the setcore exactly corresponds to the set of outcomes of strong Nash equilibria of the associated claim game correspondence.

Theorem 10.3.8 Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be an upper cycle free effectivity function and R_N a profile on A. Let $G(E) := (X_1, \ldots, X_n, A, \pi)$ be the associated claim game correspondence. Then the following two assertions hold.

(i) If σ_N is a strong Nash equilibrium of $G(E)$ at R_N, then $\pi(\sigma_N)$ is an element of the setcore of E at R_N.

(ii) If E is superadditive, then for each setcore element B of E at R_N there exists a strong Nash equilibrium σ_N of $G(E)$ at R_N such that $\pi(\sigma_N) = B$.

Proof. (i) Let $\sigma_N \in X_N$ be a strong Nash equilibrium of $G(E)$ at R_N. Define $B := \pi(\sigma_N)$. Suppose there are $S \in \mathcal{P}_0(N)$ and $B' \in E(S)$ such that $B' \tilde{P}_S B$. Define $\tau_S \in X_S$ by $\tau_i := (S, B')$ for all $i \in S$. Then $\pi(\tau_S, \sigma_{N\setminus S}) \subset B'$ and therefore $B \setminus \pi(\tau_S, \sigma_{N\setminus S}) \neq \emptyset$. Moreover, we have $F(\tau_S, \sigma_{N\setminus S}) \setminus \{B'\} \subset F(\sigma_N)$. Hence, $\pi(\tau_S, \sigma_{N\setminus S}) = \bigcap\{D \mid D \in F(\tau_S, \sigma_{N\setminus S})\} \supset \bigcap\{D \mid D \in F(\sigma_N)\} \cap B' = B \cap B'$. Since $B \setminus \pi(\tau_S, \sigma_{N\setminus S}) \neq \emptyset$, $B \cap B' \subset \pi(\tau_S, \sigma_{N\setminus S}) \subset B'$, and $B' \tilde{P}_S B$, it follows by definition of $\tilde{P}_S$ that $\pi(\tau_S, \sigma_{N\setminus S}) \tilde{P}_S B$. This leads to a contradiction since σ_N is a strong Nash equilibrium of $G(E)$ at R_N.
(ii) Let E be superadditive and let B be a setcore element of E at R_N. Define $\sigma_N \in X_N$ by $\sigma_i := (N, B)$ for all $i \in N$. Then $\pi(\sigma_N) = B$. Suppose that σ_N is not a strong Nash equilibrium of $G(E)$ at R_N. Then there exist $S \in \mathcal{P}_0(N)$ and $\tau_S \in X_S$ such that $\pi(\tau_S, \sigma_{N\setminus S}) \tilde{P}_S \pi(\sigma_N)$. Hence, $\pi(\tau_S, \sigma_{N\setminus S}) \neq A$ and there are disjoint subsets $S_1, \ldots, S_k$ of S and $B_r \in E(S_r)$ for all $r \in \{1, \ldots, k\}$ such that $\tau_j = (S_r, B_r)$ for all $j \in S_r$ and all $r \in \{1, \ldots, k\}$, and $\pi(\tau_S, \sigma_{N\setminus S}) = \bigcap_{r=1}^k B_r$. Since E is superadditive, we have $\bigcap_{r=1}^k B_r \in E(\bigcup_{r=1}^k S_r) \subset E(S)$. This leads to a contradiction since B is a setcore element of E at R_N. □

Combining Theorem 10.3.6 and Theorem 10.3.8, we obtain the following analogue of Theorem 8.5.3.

Theorem 10.3.9 Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be a superadditive and A-monotonic effectivity function and let $R_N \in \mathcal{R}_N$ a profile. Then there exists a strongly consistent game correspondence G such that

(i) $E = E^G_\alpha = E^G_\beta$

(ii) $Setcore(E, R_N) = \pi(SNE(G, R_N))$.

10.4 Toward game forms

In the previous section every upper cycle free effectivity function E is associated with a claim game correspondence $G(E) = (X_1, \ldots, X_n, A, \pi)$. Here π is a correspondence from X_N to A. In this section we derive a *claim game form* $H(E) = (X_1, \dots, X_n, A, \rho)$ from $G(E)$, where ρ is a surjective function from X_N to A. Moreover, this is done in such a way that $E_\alpha^{G(E)} = E_\alpha^{H(E)}$.

In his proof of Theorem 8.5.1, Moulin (1983) also describes a process to go from a game correspondence G to a game form H such that $E_\alpha^G = E_\alpha^H$, but this construction only works for a finite set of alternatives, while the construction described in this section can also be applied to an infinite set of alternatives. As a result this section yields an alternative proof of Theorem 8.5.1.

An obvious way to go from a game correspondence to a game form is by means of a choice function.

Lemma 10.4.1 Let $C : \mathcal{P}_0(A) \to A$ be a 'choice function', i.e., $C(B) \in B$ for all $B \in \mathcal{P}_0(A)$. Let $G = (X_1, \dots, X_n, A, \pi)$ be a game correspondence. Define $\rho = C \circ \pi$. Then

(i) $H = (X_1, \dots, X_n, A, \rho)$ is a game form

(ii) $E_\alpha^G \subset E_\alpha^H$ and $E_\beta^G \subset E_\beta^H$.

Proof. (i) The surjectiveness of ρ follows from the non-imposedness of π.
(ii) Let $S \in \mathcal{P}_0(N)$ and $B \in \mathcal{P}_0(A)$ and suppose $B \in E_\alpha^G(S)$. Then there exists a strategy $\sigma_S \in X_S$ such that $\pi(\sigma_S, \tau_{N\setminus S}) \subset B$ for all $\tau_{N\setminus S} \in X_{N\setminus S}$. But then $C \circ \pi(\sigma_S, \tau_{N\setminus S}) \in B$ for all $\tau_{N\setminus S} \in X_{N\setminus S}$. Hence $B \in E_\alpha^H(S)$.
The proof of the second assertion is similar. □

However, in general the inclusions in Lemma 10.4.1 (ii) need not be equalities, not even if G is a claim game correspondence. The following example shows that there are claim game correspondences such that for every choice function these inclusions are not equalities.

Example 10.4.2 Let $N = \{1, 2\}$, $A = \{a, b\}$, and define the effectivity function $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ as follows. $E(\{1\}) = E(\{2\}) = \{A\}$, and $E(N) = \mathcal{P}_0(A)$. Then $G(E) = (X_1, X_2, A, \pi)$, where for all $i \in N$
$X_i = \{(\{i\}, A)\} \cup \{(N, B) \mid B \in \mathcal{P}_0(A)\}$ and for all $\sigma_N \in X_N$

$$\pi(\sigma_N) = \begin{cases} \{a\} & \text{if } \sigma_1 = \sigma_2 = (N, \{a\}) \\ \{b\} & \text{if } \sigma_1 = \sigma_2 = (N, \{b\}) \\ A & \text{otherwise.} \end{cases}$$

Let $C : \mathcal{P}_0(A) \to A$ be a choice function. Suppose, without loss of generality, that $C(A) = \{a\}$. Let $\rho := C \circ \pi$ and $H(E) := (X_1, X_2, A, \rho)$. Then
$\{A\} = E_\beta^{G(E)}(\{i\}) = E_\alpha^{G(E)}(\{i\}) \subsetneqq E_\alpha^{H(E)}(\{i\}) = E_\beta^{H(E)}(\{i\}) = \{\{a\}, A\}$ for all $i \in N$.

In order to establish an equality between $E_\alpha^{G(E)}$ and $E_\alpha^{H(E)}$ for some game form $H(E)$ derived from $G(E)$, we must go beyond the scope of choice functions. In particular, $\rho(\sigma_N)$ will have to depend on σ_N itself, not just on $\pi(\sigma_N)$.
Let $C : \mathcal{P}_0(A) \to A$ be a choice function. For each $B \in \mathcal{P}_0(A)$ define a surjection h_B^C from A to B by

$$h_B^C(b) := \begin{cases} b & \text{if } b \in B \\ C(B) & \text{if } b \in A \setminus B. \end{cases} \tag{10.1}$$

Let $\overline{\triangle} : \mathcal{P}_0(A) \times \mathcal{P}_0(A) \to \mathcal{P}_0(A)$ be a binary operation on the non-empty subsets of A defined for all $B, D \in \mathcal{P}_0(A)$ as

$$B \overline{\triangle} D := \begin{cases} B & \text{if } B = D \\ (B \setminus D) \cup (D \setminus B) & \text{if } B \neq D. \end{cases}$$

If $B \neq D$, then $B \overline{\triangle} D$ is the symmetric difference between B and D. Let $B, D \in \mathcal{P}_0(A)$ with $B \neq D$. Note that $(B \overline{\triangle} B) \overline{\triangle} D = B \overline{\triangle} D$ and that $B \overline{\triangle} (B \overline{\triangle} D) = D$. So, $\overline{\triangle}$ is not associative. In order to avoid parentheses it is necessary to define the order in which a sequence of $\overline{\triangle}$ operations must be evaluated. Let $D_1, \ldots, D_k$ be elements of $\mathcal{P}_0(A)$. Then $D_1 \overline{\triangle} D_2 \overline{\triangle} D_3$ means $(D_1 \overline{\triangle} D_2) \overline{\triangle} D_3$ and for all $3 < t \leq k$ $D_1 \overline{\triangle} D_2 \overline{\triangle} \ldots \overline{\triangle} D_t$ means $(D_1 \overline{\triangle} \ldots \overline{\triangle} D_{t-1}) \overline{\triangle} D_t$. So, the evaluation of $D_1 \overline{\triangle} \ldots \overline{\triangle} D_t$ is from left to right.

Proposition 10.4.3 Let $2 \leq k$ and $1 \leq t \leq k$. Let $D_1, \ldots, D_k$ be elements of $\mathcal{P}_0(A)$. Then $\{D_1 \overline{\triangle} \ldots \overline{\triangle} D_{t-1} \overline{\triangle} B \overline{\triangle} D_{t+1} \overline{\triangle} \ldots \overline{\triangle} D_k \mid B \in \mathcal{P}_0(A)\} = \mathcal{P}_0(A)$.

Proof. Let $D, D' \in \mathcal{P}_0(A)$. Then there is a $B \in \mathcal{P}_0(A)$ such that $D \overline{\triangle} B = D'$: When $D = D'$, then take $B = D$, else take $B = D \triangle D'$.
So, $\{D \overline{\triangle} B \mid B \in \mathcal{P}_0(A)\} = \{B \overline{\triangle} D \mid B \in \mathcal{P}_0(A)\} = \mathcal{P}_0(A)$.
Consequently, $\{D_1 \overline{\triangle} \ldots \overline{\triangle} D_{t-1} \overline{\triangle} B \mid B \in \mathcal{P}_0(A)\} = \mathcal{P}_0(A)$. Hence,
$\{D_1 \overline{\triangle} \ldots \overline{\triangle} D_{t-1} \overline{\triangle} B \overline{\triangle} D_{t+1} \mid B \in \mathcal{P}_0(A)\} = \{P \overline{\triangle} D_{t+1} \mid P \in \mathcal{P}_0(A)\} = \mathcal{P}_0(A)$.
Hence,
$\{D_1 \overline{\triangle} \ldots \overline{\triangle} D_{t-1} \overline{\triangle} B \overline{\triangle} D_{t+1} \overline{\triangle} D_{t+2} \mid B \in \mathcal{P}_0(A)\} = \{P \overline{\triangle} D_{t+2} \mid P \in \mathcal{P}_0(A)\} = \mathcal{P}_0(A)$.
Repetition of this argument yields

$\{D_1\overline{\triangle}\ldots\overline{\triangle}D_{t-1}\overline{\triangle}B\overline{\triangle}D_{t+1}\overline{\triangle}\ldots\overline{\triangle}D_k \mid B \in \mathcal{P}_0(A)\} = \mathcal{P}_0(A).$ □

Now we are able to define a claim game form derived from a claim game correspondence.

Let $G(E) = (X_1, \ldots, X_n, A, \pi)$ be a claim game correspondence associated with an upper cycle free effectivity function E. Let $C : \mathcal{P}_0(A) \to A$ be a choice function and let $\{h_B^C \mid B \in \mathcal{P}_0(A)\}$ be as defined in (10.1).

Define $f_E : X_N \to \mathcal{P}_0(A)$ as follows.

For $\sigma_N = (S_i, B_i)_{i\in N} \in X_N$, $f_E(\sigma_N) := B_1\overline{\triangle}\ldots\overline{\triangle}B_n$. Then the *claim game form* $H(E) := (X_1, \ldots, X_n A, \rho)$ *corresponding to* $G(E)$ *and* C is defined by

$$\rho(\sigma_N) := h^C_{\pi(\sigma_N)}(C \circ f_E(\sigma_N))$$

for all $\sigma_N \in X_N$. Clearly, $\rho(\sigma_N) \in \pi(\sigma_N)$ for all $\sigma_N \in X_N$. So by the non-imposedness of π it follows that ρ is surjective.

We now show that the α-effectivity functions of $G(E)$ and $H(E)$ coincide for every upper cycle free effectivity function E.

Theorem 10.4.4 Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be an upper cycle free effectivity function. Let $G(E) = (X_1, \ldots, X_n, A, \pi)$ be the claim game correspondence associated with E, and let $H(E) = (X_1, \ldots, X_n, A, \rho)$ be the claim game form corresponding to $G(E)$ and a choice function C. Then

$$E_\alpha^{G(E)} = E_\alpha^{H(E)}.$$

Proof. As in the proof of Lemma 10.4.1 (ii) one can show that $E_\alpha^{G(E)} \subset E_\alpha^{H(E)}$. It remains to prove that $E_\alpha^{G(E)}(S) \supset E_\alpha^{H(E)}(S)$ for all $S \in \mathcal{P}_0(N)$. Let $S \in \mathcal{P}_0(N)$ and $B \in E_\alpha^{H(E)}(S)$. We have to prove that $B \in E_\alpha^{G(E)}(S)$. This is obvious if $B = A$ or if $S = N$. Therefore, suppose $B \neq A$ and $S \neq N$. Since $B \in E_\alpha^{H(E)}(S)$, there is a strategy $\sigma_S \in X_S$ such that for all $\tau_{N\setminus S} \in X_{N\setminus S}$ we have $\rho(\sigma_S, \tau_{N\setminus S}) \in B$.

Claim: For each $D \in \mathcal{P}_0(A)$ there is an $i \in S$ such that $\sigma_i \neq (N, D)$.

Proof of the claim: Let $D \in \mathcal{P}_0(A)$. Suppose for all $i \in S$, $\sigma_i = (N, D)$. Take $a \in A \setminus B$. By Proposition 10.4.3 it follows that there are $D_j \in \mathcal{P}_0(A)$, $j \in N \setminus S$, such that $D_1\overline{\triangle}D_2\overline{\triangle}\ldots\overline{\triangle}D_n = \{a\}$, where for ease of notation $D_i = D$ if $i \in S$. Let $\tau_j = (N, D_j)$ for all $j \in N \setminus S$. Then

$\rho(\sigma_S, \tau_{N\setminus S}) = h^C_{\pi(\sigma_S,\tau_{N\setminus S})}(C \circ f_E(\sigma_S, \tau_{N\setminus S})) = h^C_{\pi(\sigma_S,\tau_{N\setminus S})}(C(\{a\})) = h^C_{\pi(\sigma_S,\tau_{N\setminus S})}(a).$

If $D_j = D$ for all $j \in N \setminus S$, then $\{a\} = D\overline{\triangle}D\overline{\triangle}\ldots\overline{\triangle}D = D$ and $\pi(\sigma_S, \tau_{N\setminus S}) = D$. However, this would imply that $\rho(\sigma_S, \tau_{N\setminus S}) = a \notin B$.

So, there is a $j \in N \setminus S$ such that $D_j \neq D$. Hence, $\pi(\sigma_S, \tau_{N\setminus S}) = A$ and again $\rho(\sigma_S, \tau_{N\setminus S}) = a \notin B$. So, there is no $D \in \mathcal{P}_0(A)$ such that for all $i \in S$ we have $\sigma_i = (N, D)$, and this proves the claim.

Fix $i \in N \setminus S$. For each $D \in \mathcal{P}_0(A)$, consider the strategy vector $\tau^D_{N\setminus S} \in X_{N\setminus S}$ defined by $\tau^D_i = (N, D)$ and $\tau^D_j = (N, A)$ for all $j \in N \setminus (S \cup \{i\})$. Then it follows that there exists a $Z \in \mathcal{P}_0(A)$ such that for all $D \in \mathcal{P}_0(A)$, $\pi(\sigma_S, \tau^D_{N\setminus S}) = Z$.

By definition $\rho(\sigma_S, \tau^D_{N\setminus S}) = h^C_Z(C \circ f_E(\sigma_S, \tau^D_{N\setminus S}))$. By Proposition 10.4.3 we have $\{f_E(\sigma_S, \tau^D_{N\setminus S}) \mid D \in \mathcal{P}_0(A)\} = \mathcal{P}_0(A)$ and therefore $\{\rho(\sigma_S, \tau^D_{N\setminus S}) \mid D \in \mathcal{P}_0(A)\} = Z$. Since $\rho(\sigma_S, \tau^D_{N\setminus S}) \in B$ for all $D \in \mathcal{P}_0(A)$, we have $\pi(\sigma_S, \tau^D_{N\setminus S}) = Z \subset B$ for all $D \in \mathcal{P}_0(A)$. But then it readily follows that $\pi(\sigma_S, \tau_{N\setminus S}) \subset B$ for all $\tau_{N\setminus S} \in X_{N\setminus S}$. Hence, $B \in E^{G(E)}_\alpha(S)$. □

Corollary 10.4.5 Let $E : \mathcal{P}_0(N) \to 2^{\mathcal{P}_0(A)}$ be an upper cycle free effectivity function. Then there exists a game form H such that $E^H_\alpha = E$ if and only if E is superadditive and A-monotonic.

Proof. Combine Theorems 10.3.6 and 10.4.4. □

Example 10.4.6 Again consider Example 10.4.2. Applying Theorem 10.4.4 yields $\rho(\sigma_N) = b$ if $\sigma_1 = \sigma_2 = (N, \{b\})$ or if both $\sigma_1 = (N, \{a\})$ and $\sigma_2 \in \{(N, A), (\{2\}, A)\}$ or if both $\sigma_2 = (N, \{a\})$ and $\sigma_1 \in \{(N, A), (\{1\}, A)\}$. In all other cases $\rho(\sigma_N) = a$. Now it follows that $E^{H(E)}_\beta(\{1\}) = E^{H(E)}_\beta(\{2\}) = \{\{a\}, A\}$. Moreover, since E is not maximal, it follows that there is no game form H such that $E^H_\beta = E$.

We do not know whether for a stable and maximal effectivity function E and a preference profile R_N, the strong Nash equilibria of the claim game form corresponding to $G(E)$ and a choice function C exactly correspond to the core of E at R_N. So it is an open problem whether the construction we presented in this section is appropriate to obtain an alternative proof of Theorem 8.5.3.

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Subject index

Samenvatting

Er zijn talloze situaties te bedenken waarin meerdere partijen met tegenstrijdige belangen betrokken zijn in een gezamenlijk beslissingsproces. Het probleem in deze situaties is dan vaak hoe een goede oplossing gevonden kan worden? In de literatuur zijn diverse wiskundige modellen geïntroduceerd om hierin een beter inzicht te verkrijgen. In de drie delen van dit proefschrift worden enkele van deze modellen bestudeerd. Het proefschrift begint met een algemene inleiding in hoofdstuk 1.

Deel I bestudeert situaties waar partijen samenwerken in een gezamenlijk project. De opbrengsten of kostenbesparingen die door de samenwerking ontstaan, dienen op een rechtvaardige manier te worden verdeeld over de deelnemende partijen. Een wiskundig instrument om dit soort problemen te analyseren wordt aangedragen door de coöperatieve speltheorie. Binnen de coöperatieve speltheorie zijn diverse modellen ontwikkeld om een goed inzicht te verkrijgen in 'samenwerkingssituaties'. Prominente modellen zijn coöperatieve spelen met/zonder zijdelingse betalingen (TU- en NTU-spelen) en onderhandelingsproblemen. Voor elk van deze modellen zijn in de literatuur diverse verdeelmethoden (oplossingsconcepten) geïntroduceerd. In de hoofdstukken 2 tot en met 5 van dit proefschrift concentreren we ons op een bepaald type oplossingsconcepten, de zogenaamde compromiswaarden. Het algemene principe dat ten grondslag ligt aan deze oplossingsconcepten is dat het deel van de gezamenlijke opbrengst dat aan een afzonderlijke partij wordt toegekend, bepaald wordt door middel van een compromis tussen een pessimistische ondergrens en een optimistische bovengrens voor deze uitbetaling.
Hoofdstuk 2 geeft een overzicht van compromiswaarden binnen de coöperatieve speltheorie. De meeste aandacht wordt besteed aan de τ-waarde voor TU-spelen, de Raiffa-Kalai- Smorodinsky (RKS) oplossing voor onderhandelingsproblemen en de compromiswaarde voor NTU-spelen. We besluiten hoofdstuk 2 met een aantal toepassingen van coöperatieve speltheorie die duidelijk het belang van compromiswaar-

den illustreren.
Een belangrijke toepassing van coöperatieve speltheorie ligt binnen kostenallocatieproblemen. In hoofdstuk 3 bestuderen we een kostenallocatiemethode, die in de dertiger jaren al werd toegepast door de Tennessee Valley Authority. Het blijkt dat deze verdeelmethode te beschouwen is als een compromiswaarde.
In de hoofdstukken 4 en 5 concentreren we ons op NTU-spelen. Hoofdstuk 4 bestudeert de compromiswaarde voor NTU-spelen. Er worden bekende resultaten uitgebreid naar een grotere klasse van spelen.
Hoofdstuk 5 introduceert een nieuw oplossingsconcept voor NTU-spelen, de MC-waarde. De MC-waarde is een uitbreiding van een bekende verdeelmethode voor TU-spelen en is eveneens te beschouwen als een compromiswaarde. Naast diverse eigenschappen van de MC-waarde geven we twee karakteriseringen van dit oplossingsconcept.

In deel II van dit proefschrift bestuderen we situaties waarin een bepaalde hoeveelheid van een (oneindig deelbaar) goed verdeeld moet worden onder een aantal economische agenten. Aangenomen wordt dat elk van deze agenten preferenties heeft over het goed die 'single-peaked' zijn, dat wil zeggen tot aan een bepaalde hoeveelheid wil een agent meer van het goed consumeren, na deze hoeveelheid geldt het omgekeerde. De hoeveelheid van het goed die door een agent het meest wordt geprefereerd, wordt de 'piek' genoemd. Het probleem is dat het totaal van de pieken van de verschillende agenten in het algemeen niet gelijk is aan de hoeveelheid van het goed die verdeeld dient te worden.
Het bovengenoemde verdeelprobleem is de afgelopen jaren uitgebreid in de literatuur bestudeerd. De eerste die dit probleem op een systematische wijze analyseerde was Sprumont (1991). Zijn hoofdresultaat toont aan dat er een unieke verdeelregel voor dit type problemen is die voldoet aan Pareto optimaliteit, anonimiteit en niet-manipuleerbaarheid. Sprumont noemde deze regel de uniforme regel. Ook uit latere analyses blijkt dat de uniforme regel beschouwd kan worden als een prominente verdeelregel voor dit soort problemen. Hoofdstuk 6 geeft naast een formele beschrijving van het model ook een overzicht van de belangrijkste resultaten uit de literatuur.
In hoofdstuk 7 worden relaties beschreven tussen de uniforme regel enerzijds en de Nash en lexicografisch egalitaire onderhandelingsoplossing anderzijds. Gebaseerd op bekende resultaten uit de theorie van onderhandelingsproblemen, worden nieuwe karakteriseringen van de uniforme regel besproken.

Deel III bestudeert sociale keuzeproblemen. Een centraal probleem in de sociale keuzetheorie is, gegeven de individuele preferenties van agenten over een bepaalde verzameling alternatieven, om een deelverzameling van alternatieven te kiezen die voor iedereen accepteerbaar is in de zin dat zij een goede afspiegeling vormt van de individuele preferenties. In de literatuur zijn diverse van dit soort keuzeregels bestudeerd. Bij een gegeven keuzeregel is het interessant om te bestuderen welke mogelijkheden individuele agenten of groepen agenten hebben om de uitkomst te beïnvloeden door het opgeven van andere dan hun werkelijke preferenties. Het is duidelijk dat door de mogelijkheden die coalities hebben om de uitkomst te manipuleren een bepaalde machtsverdeling binnen de gemeenschap ontstaat. Deze collectieve machtsverdeling kan worden gemodelleerd door gebruik te maken van effectiviteitsfunkties. Een effectiviteitsfunktie specificeert voor elke coalitie de collectie van deelverzamelingen van alternatieven waarvoor deze coalitie 'effectief' is in de zin dat zij kan garanderen dat de uiteindelijke uitkomst binnen een zo'n verzameling ligt. In de hoofdstukken 8, 9 en 10 bestuderen we effectiviteitsfunkties.

Hoofdstuk 8 geeft een overzicht van de meest belangrijke resultaten uit de literatuur over effectiviteitsfunkties.

In hoofdstuk 9 introduceren we een nieuwe klasse van effectiviteitsfunkties die diverse bestaande klassen generaliseert. Voorts worden diverse eigenschappen van de effectiviteitsfunkties binnen deze klasse bestudeerd.

Hoofdstuk 10 onderzoekt relaties tussen effectiviteitsfunkties en (niet-coöperatieve) spelkiemen. In tegenstelling tot de bestaande literatuur kijken we hierbij niet naar spelkiemen waar de uitbetalingsafbeelding een funktie is, maar waar deze een correspondentie (multifunktie) is. We laten onder andere zien dat enkele van de resultaten die in hoofdstuk 8 besproken zijn, uitgebreid kunnen worden naar grotere klassen.

Curriculum vitae

De schrijver van dit proefschrift werd geboren op 28 november 1967 te Nijmegen. Na in 1972 te zijn verhuisd naar Arnhem, volgde hij van 1980 tot 1986 het voorbereidend wetenschappelijk onderwijs aan het Nederrijn College te Arnhem.
In september 1986 begon hij met de studie wiskunde aan de Katholieke Universiteit Nijmegen. Het propaedeutisch examen werd afgelegd in augustus 1987. Het doctoraal examen met als hoofdrichtingen speltheorie, operations research en statistiek werd behaald in maart 1991.
Per 1 april 1991 trad hij in dienst als AIO bij de vakgroep Econometrie van de Katholieke Universiteit Brabant, waar hij onder de stimulerende leiding van prof. dr. S.H. Tijs en dr. P.E.M. Borm gedurende vier jaar onderzoek verrichtte op het gebied van de speltheorie. De meeste resultaten van dit onderzoek zijn weergegeven in dit proefschrift.
Naast zijn baan als AIO was hij van september 1993 tot mei 1994 werkzaam bij het Gemeentelijk Havenbedrijf Rotterdam. Samen met medewerkers van het Gemeentelijk Havenbedrijf Rotterdam en enkele stagiaires onderzocht hij of speltheorie een rol kan spelen om samenwerking tussen bedrijven binnen de Rotterdamse haven te bevorderen.